An Optimal Parallel Algorithm for Hinge Vertex Probrem of a Circular-Arc Graph

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Abstract: Let $G = (V, E)$ be an undirected simple graph with $u \in V$. If there exist any two vertices in G whose distance becomes longer when a vertex u is removed, then u is defined as a hinge vertex. Finding the set of hinge vertices in a graph is useful for identifying critical nodes in an actual network. A number of studies concerning hinge vertices have been made in recent years. In a number of graph problems, it is known that more efficient sequential or parallel algorithms can be developed by restricting classes of graphs. In this paper, we shall propose a parallel algorithm which runs in $O(\log n)$ time with $O(n/\log n)$ processors on EREW PRAM for finding all hinge vertices of a circular-arc graph.

Key words: Parallel algorithms, Circular-arc graphs, Hinge vertices

1 Introduction

Given a simple undirected graph $G = (V, E)$ with vertex set V and edge set E. Let $G - u$ be a subgraph induced by the vertex set $V - \{u\}$. The distance $dis_G(x, y)$ is defined as the length (i.e., the number of edges) of the shortest path between vertices x and y in G. Chang et al. defined that $u \in V$ is a *hinge vertex* if there exist two vertices $x, y \in V - \{u\}$ such that $dis_{G-u}(x, y) > dis_G(x, y)$ [1]. A graph without hinge vertices is called a *self-repairing graph*. Farley and Proskurowski presented a constructive characterization related to the class of self-repairing graphs [2]. For the design and analysis of distributed networks, the analysis of topological properties is a very important research topic. The overall cost of communication in networks is increased if a computer corresponding to a hinge vertex stalls. Therefore, finding the set of hinge vertices in a graph is useful for identifying critical nodes in an actual network. A number of studies concerning hinge vertices have been made in recent years. A trivial $O(n^3)$ sequential algorithm for finding all hinge vertices of a simple graph is straightforwardly obtained by a result in [1], e.g., Theorem 1 in this paper. Furthermore, Theorem 1 leads to NC algorithms as well as efficient sequential algorithms for finding hinge vertices in several graphs. In a number of graph problems, it is well-known that more efficient sequential or parallel algorithms have been developed by restricting classes of graphs. For instance, Chang et al. presented an $O(n + m)$ time algorithm for finding all hinge vertices of a *strongly chordal graph* [1]. Ho et al. presented a linear time algorithm for finding all hinge vertices of a *permutation graph* [3]. We have presented a parallel algorithm, which runs in $O(\log n)$ time with $O(n)$ processors on CREW (Concurrent-Read Exclusive-Write Parallel Random Access Machine), for finding all hinge vertices of an *interval graph* [4] and a *trapezoid graph* [5], respectively. Recently, an optimal parallel algorithm for interval graphs was proposed by Hsu et al. [6] which runs in $O(\log n)$ time with $O(n/\log n)$ processors. However, no efficient parallel algorithm for finding all hinge vertices of a *circular-arc graph* [7] has been presented. Problems that can be solved efficiently for interval graphs or trapezoid graphs may not always be solvable efficiently for circular-arc graphs. For example, the coloring problem can be solved efficiently for interval graphs (see e.g., [8] for a sequential algorithm and [9] for a parallel algorithm) or trapezoid graphs (see e.g., [10] for a sequential algorithm and [11] for a parallel algorithm), however, it is NP-hard for circular-arc graphs [12]. Fortunately, the problem of finding all hinge vertices is also solvable efficiently for a circular-arc graph, as will be seen in this paper. We shall propose an optimal parallel algorithm which runs in $O(\log n)$ time with $O(n/\log n)$ processors on EREW PRAM (Exclusive-Read Exclusive-Write Parallel Random Access Machine) for finding all hinge vertices of a circular-arc graph.

2 Definition

We first illustrate the *circular-arc model* before defining the circular-arc graph. Consider a unit circle C and a family A of n circular-arcs $A_1, A_2, \ldots, A_n$ along the circumference of C. Each circular-arc A_i has two endpoints, *left endpoint* a_i and *right endpoint* b_i, respectively, and is denoted by $A_i = [a_i, b_i]$. The left endpoint a_i (resp., right endpoint b_i) is the last point

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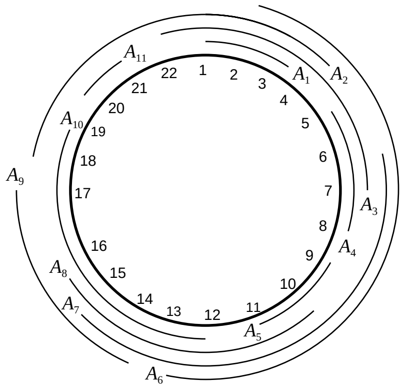


Figure 1: Circular-arc model CM.

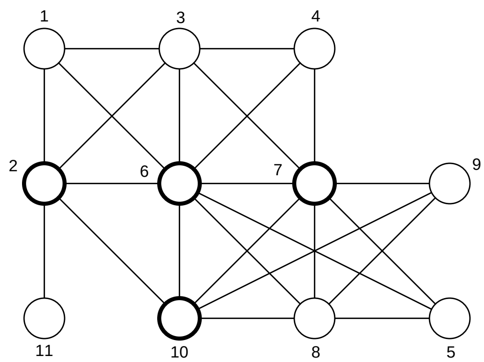


Figure 2: Circular-arc graph G.

of A_i that we encounter when walking along A_i counterclockwise (resp., clockwise). Without loss of generality, the coordinates of all left and right endpoints are distinct and are assigned clockwise with consecutive integer values $1, 2, \ldots, 2n$. The circular-arc numbers i, j are assigned to each circular-arc in increasing order of their right endpoints b_i's, i.e., $A_i < A_j$ if $b_i < b_j$. Note that circular-arc A_i with $a_i > b_i$ is called a *feedback circular-arc*. The geometric representation described above is called a *circular-arc model*. Figure 1 illustrates a circular-arc model CM, consisting of eleven circular-arcs (note that A_2 and A_3 are feedback circular-arcs).

A graph $G = (V, E)$ is called a *circular-arc graph* if there exists a family of circular-arcs $A = \{A_1, A_2, \ldots, A_n\}$ such that there is a one-to-one correspondence between vertex $i \in V$ and the circular-arc $A_i \in A$ in such a way that an edge $(i, j) \in E$ if and only if A_i intersects with A_j in CM. Figure 2 illustrates the circular-arc graph G corresponding to CM shown in Fig. 1. In this example, all hinge vertices of G are vertices 2, 6, 7 and 10.

We next introduce an *extended circular-arc model* constructed from a CM for making the problem easier. We first cut CM at endpoint 1 and next unroll onto the real horizontal line. Each circular-arc $A_i = [a_i, b_i]$ in CM is also changed to horizontal line segment $I_i = [a_i, b_i]$ called *interval* by executing the above process. Also, $I_{i+n} = [a_i + 2n, b_i + 2n]$ is defined as a *dummy interval* of I_i, and is often denoted by $I_{d(i)}$ for convenience. When some $A_i = [a_i, b_i]$ is a feedback circular-arc in CM, circular-arc $A_i = [a_i, b_i]$ is changed to interval $I_i = [a_i - 2n, b_i]$, and a dummy interval $I_{i+n}(I_{d(i)}) = [a_i, b_i + 2n]$ is added on ECM. Figure 3 shows the extended circular-arc model ECM constructed from the circular-arc model CM illustrated in Fig. 1. When no feedback circular-arc exists in CM, ECM is equivalent to the *interval model* [8], that is, the circular-arc graph G corresponding to CM belongs to the class of interval graphs. In this case, this problem can be solved by applying the algorithm for finding all hinge vertices of an interval graph [6].

In what follows, we define some functions and terms used in this paper. For I_i in ECM, $M(i)$ is the interval number j such that interval $I_j (j \geq i)$ is the largest one intersecting with I_i. Similarly, for I_i in ECM, $SM(i)$ is the interval number j such that $I_j (j \geq i)$ of the second largest one intersecting with I_i. When such interval I_j does not exist, let $M(i) = i$ and $SM(i) = i$, respectively. Formally, $M(i) = \max\{j \mid I_j \text{ contains } b_i\}$ and $SM(i) = \max\{i, \text{smax}\{j \mid I_j \text{ contains } b_i\}\}$, where, smax is the function that denotes the second largest element in a set. Also, for $1 \leq i \leq n$, we define $D(i) = \{k \mid b_{SM(i)} < k < b_{M(i)}, k \in \mathcal{N}\}$. Table 1 shows $M(i)$, $SM(i)$ and $D(i)$ for ECM illustrated in Fig. 3.

In the followings, we define $MV = \{M(i) \mid D(i) \neq \emptyset, 1 \leq i \leq n\}$, where MV has no multiple element. In the example shown in Table 1, $MV = \{6, 7, 10, 13\}$. Also, we define $MV_j = \{i \mid M(i) = j, 1 \leq i \leq n\}$ for all $j \in MV$. For instance, $MV_6 = \{1, 2\}$, $MV_7 = \{3, 4\}$, $MV_{10} = \{6, 7, 8, 9\}$ and $MV_{13} = \{10, 11\}$ in Table 1. For all $j \in MV$, *represent vertex* R_j of MV_j is defined as the minimum value element of MV_j, that is, $R_j = \min\{i \mid i \in MV_j\}$, $j \in MV$. Moreover, *represent vertex set* (RVS) is defined as a set consisting of all represent vertices R_j, that is, $RVS = \{R_j \mid j \in MV\}$. In the example of Table 1, $R_6 = 1$, $R_7 = 3$, $R_{10} = 6$ and $R_{13} = 10$ for MV_6, MV_7, MV_{10} and MV_{13}, respectively, and $RVS = \{1, 3, 6, 10\}$. Also, we define $D_{RVS} = \cup_{j \in RVS} D(j)$, that is, $D_{RVS} = D(1) \cup D(3) \cup D(6) \cup D(10) = \{8, 9, 10, 11, 12, 14, 17, 18, 20, 21, 22, 23, 24, 25\}$.

Next, for $k \in D_{RVS}$, $la(k)$ and $ra(k)$ are defined as follows. $la(k) = i$ when there exists some $i (1 \leq i \leq n)$ that satisfies $a_i + 2n = k$. $ra(k) = i$ when there exists some $i (1 \leq i \leq n)$ that satisfies $a_i = k$. In the example of Table 1, $la(18) = 2$ since $a_i + 2n = k$ for $n = 11$, $i = 2$ and $k = 18 \in D_{RVS}$. $ra(12) = 10$ since $a_i = k$ for $i = 10$ and $k = 12 \in D_{RVS}$.

Moreover, for $k \in D_{RVS}$, $p(k)$ is defined as follows. $p(k) = m$ where a_m is the maximum value of all a_i satisfying $k \in D(i)$. Formally, $p(k) = m$

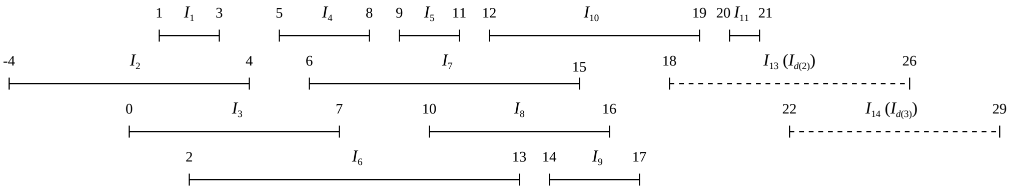


Figure 3: Extended circular-arc model ECM.

Table 1: Details of arrays $M(i)$, $SM(i)$, $D(i)$ for ECM illustrated in Fig. 3.

i	1	2	3	4	5	6	7	8	9	10	11	13($d(2)$)	14($d(3)$)
a_i	1	-4	0	5	9	2	6	10	14	12	20	18	22
b_i	3	4	7	8	11	13	15	16	17	19	21	26	29
$M(i)$	6	6	7	7	8	10	10	10	10	13($d(2)$)	13($d(2)$)	14($d(3)$)	14($d(3)$)
$SM(i)$	3	3	6	6	7	8	9	9	9	10	11	13($d(2)$)	14($d(3)$)
$D(i)$	8,..,12	8,..,12	14	14	$\emptyset$	17,18	18	18	18	20,..,25	22,..,25	-	-

iff $a_m = \max\{a_i \mid k \in D(i)\}$. In the example of Table 1, $p(18) = 9$ for $k = 18 \in D_{RVS}$, since $18 \in D(6) \cap D(7) \cap D(8) \cap D(9)$ and a_9 is the maximum value among $a_6 = 2$, $a_7 = 6$, $a_8 = 10$ and $a_9 = 14$. Table 2 shows details of arrays of $la(k), ra(k)$ and $p(k)$ for $k \in D_{RVS}$.

3 Some Properties on the Hinge Vertices

Theorem 1 [1] due to Chang et al. characterizes the hinge vertices of a simple graph. We apply this theorem for finding efficiently the hinge vertices of a given circular-arc graph.

Theorem 1 *For a graph $G = (V, E)$, a vertex $u \in V$ is a hinge vertex of G if and only if there exist two nonadjacent vertices x, y such that u is the only vertex adjacent with both x and y in G.* $\Box$

For simplicity, we say, throughout this paper, that u is the hinge vertex for x and y when the condition in Theorem 1 is satisfied. The following corollary 1 is derived immediately from Theorem 1, therefore the proof is omitted.

Corollary 1 *Let CM be a circular-arc model, and let $G = (V, E)$ be a circular-arc graph corresponding to CM. We assume that vertices $u, x, y (x < y) \in V$ in G correspond to circular-arcs $A_u, A_x, A_y (x < y)$ in CM, respectively. Then, a vertex u is a hinge vertex for x and y in G if and only if A_x does not intersect with A_y, and A_u is the only circular-arc intersecting with both A_x and A_y in CM.* $\Box$

Corollary 2 *Let CM be a circular-arc model, and let $G = (V, E)$ be a circular-arc graph corresponding to CM. We assume that vertices $u, x, y (x < y) \in V$ in G correspond to circular-arcs $A_u, A_x, A_y (x < y)$ in CM, respectively. ECM is an extended circular-arc model constructed from CM. Furthermore, the interval $I_{d(x)} = [a_x + 2n, b_x + 2n]$ is a dummy interval of I_x in ECM. Then, a vertex u is a hinge vertex for x and y of G if and only if neither I_x nor $I_{d(x)}$ intersects with I_y, and at least one of the following four conditions holds in ECM.*

1. *I_u is the only interval intersecting with both I_x and I_y, and there exists no interval (except I_u or $I_{d(u)}$) intersecting with both I_y and $I_{d(x)}$.*
2. *$I_{d(u)}$ is the only interval intersecting with both I_x and I_y, and there exists no interval (except I_u or $I_{d(u)}$) intersecting with both I_y and $I_{d(x)}$.*
3. *I_u is the only interval intersecting with both I_y and $I_{d(x)}$, and there exists no interval (except I_u or $I_{d(u)}$) intersecting with both I_x and I_y.*
4. *$I_{d(u)}$ is the only interval intersecting with both I_y and $I_{d(x)}$, and there exists no interval (except I_u or $I_{d(u)}$) intersecting with both I_x and I_y.* $\Box$

In the followings, we shall describe lemmas characterizing hinge vertices in a circular-arc graph. Hereafter, for simplicity, we often denote $I_x < I_y$ the relation between two intervals I_x and I_y with $x < y$ when no confusion may arise.

Lemma 1 *Let CM be a circular-arc model, and let $G = (V, E)$ be a circular-arc graph with $u, x, y (x < y) \in V$ corresponding to CM. Also, let ECM be an extended circular-arc model constructed from CM. Assume that u is a hinge vertex for x and y of G. Then, at least one of the following four cases holds. $I_u = I_{M(x)}$, $I_{d(u)} = I_{M(x)}$, $I_u = I_{M(y)}$ and $I_{d(u)} = I_{M(y)}$ in ECM.*

(Proof) At least one of the conditions 1, 2, 3 and 4 of Corollary 2 is satisfied since u is a hinge vertex for x and y of G. Now, suppose that condition 1 of

Table 2: Details of arrays $la(k)$, $ra(k)$, $p(k)$ for $k \in D_{RVS}$.

$k \in D_{RVS}$	8	9	10	11	12	14	17	18	20	21	22	23	24	25
$la(k)$	*	*	*	*	*	*	*	2	*	*	3	1	6	*
$ra(k)$	*	5	8	*	10	9	*	*	11	*	*	*	*	*
$p(k)$	1	1	1	1	1	4	6	9	10	10	11	11	11	11

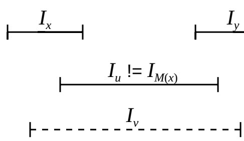


Figure 4: Example for Lemma 1.

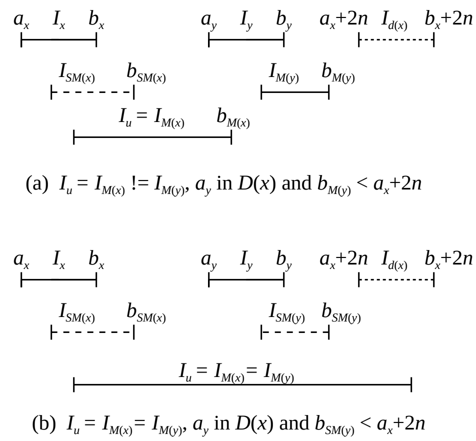


Figure 5: Illustration of Lemmas 2 and 3.

Corollary 2 holds, that is, neither I_x nor $I_{d(x)}$ intersect with I_y, I_u is the only interval intersecting with both I_x and I_y, and there is no interval (except I_u or $I_{d(u)}$) intersecting with both I_y and $I_{d(x)}$ in ECM. Assume, to the contrary, $I_u \neq I_{M(x)}$ in ECM. This means that there exists some $I_v(u < v)$ intersecting with both I_x and I_y. This, however, contradicts the fact that I_u is the only interval intersecting with both I_x and I_y. Thus, $I_u = I_{M(x)}$ when condition 1 of Corollary 2 holds (see Fig. 4).

Similarly, $I_{d(u)} = I_{M(x)}$, $I_u = I_{M(y)}$ and $I_{d(u)} = I_{M(y)}$ hold when conditions 2, 3 and 4 of Corollary 2 are satisfied, respectively. Therefore, at least one of the following four cases. $I_u = I_{M(x)}$, $I_{d(u)} = I_{M(x)}$, $I_u = I_{M(y)}$ and $I_{d(u)} = I_{M(y)}$ in ECM when u is a hinge vertex for x and y of G. □

Lemma 2 *Let CM be a circular-arc model, and let $G = (V, E)$ be a circular-arc graph with $u, x, y (x < y) \in V$ corresponding to CM. Also, let ECM be an extended circular-arc model constructed from CM. Assume that u is a hinge vertex for x and y of G. Then, at least one of the following four conditions holds in ECM.*

1. *$I_u = I_{M(x)}$, $a_y \in D(x)$ and if $I_{M(x)} \neq I_{M(y)}$ then $b_{M(y)} < a_x + 2n$ otherwise $b_{SM(y)} < a_x + 2n$.*
2. *$I_{d(u)} = I_{M(x)}$, $a_y \in D(x)$ and if $I_{M(x)} \neq I_{M(y)}$ then $b_{M(y)} < a_x + 2n$ otherwise $b_{SM(y)} < a_x + 2n$.*
3. *$I_u = I_{M(y)}$, $a_x + 2n \in D(y)$ and if $I_{M(y)} \neq I_{M(d(x))}$ then $b_{M(x)} < a_y$ otherwise $b_{SM(x)} < a_y$.*
4. *$I_{d(u)} = I_{M(y)}$, $a_x + 2n \in D(y)$ and if $I_{M(y)} \neq I_{M(d(x))}$ then $b_{M(x)} < a_y$ otherwise $b_{SM(x)} < a_y$.*

(Proof) At least one of the conditions 1, 2, 3 and 4 of Corollary 2 is satisfied since u is a hinge vertex for x and y of G. We show that condition 1 of Lemma 2 holds if condition 1 of Corollary 2 is satisfied. Suppose that condition 1 of Corollary 2 holds, that is, neither I_x nor $I_{d(x)}$ intersect with I_y, I_u is the only interval intersecting with both I_x and I_y, and there is no interval (except I_u or $I_{d(u)}$) intersecting with both I_y and $I_{d(x)}$ in ECM. $b_x < a_y$ is immediately derived since I_x does not intersect with I_y. Also, $a_y < b_{M(x)}$ since I_u intersects with both I_x and I_y, and $I_u = I_{M(x)}$ by Lemma 1. Next, to the contrary, assume that $b_{SM(x)} > a_y$, then $I_{SM(x)}$ intersects with both I_x and I_y. This, however, contradicts the fact that I_u is the only interval intersecting with both I_x and I_y. Thus, $b_{SM(x)} < a_y < b_{M(x)}$, i.e., $a_y \in D(x)$. In what follows, consider the case of $I_{M(x)} \neq I_{M(y)}$ (see Fig. 5-(a)). Suppose that there exists dummy interval $I_{d(x)} = [a_x + 2n, b_x + 2n]$ of I_x. If $b_{M(y)} > a_x + 2n$, $I_{M(y)}$ intersects with both I_y and $I_{d(x)}$. This contradicts condition 1 of Corollary 2 that there is no interval (except I_u or $I_{d(u)}$) intersecting with both I_y and $I_{d(x)}$, hence, $b_{M(y)} < a_x + 2n$. Next, consider the case of $I_{M(x)} = I_{M(y)}$ (see Fig. 5-(b)). If $b_{SM(y)} > a_x + 2n$, $I_{SM(y)}$ intersects with both I_y and $I_{d(x)}$. This contradicts condition 1 of Corollary 2 that there is no interval (except I_u or $I_{d(u)}$) intersecting with both I_y and $I_{d(x)}$, hence, $b_{SM(y)} < a_x + 2n$. Therefore, condition 1 of Lemma 2, "$I_u = I_{M(x)}$, $a_y \in D(x)$ and if $I_{M(x)} \neq I_{M(y)}$ then $b_{M(y)} < a_x + 2n$ otherwise $b_{SM(y)} < a_x + 2n$" in ECM holds if condition 1 of Corollary 2 is satisfied.

In a similar manner, conditions 2, 3 and 4 of Lemma 2 hold when conditions 2, 3 and 4 of Corollary 2 are satisfied, respectively. □

Lemma 3 *Let CM be a circular-arc model, and let $G = (V, E)$ be a circular-arc graph with $u, x, y (x < y) \in V$ corresponding to CM. Also, let ECM be an extended circular-arc model constructed from CM. Vertex u is a hinge vertex for x and y of G if and only* *if at least one of the following four conditions holds in ECM.*

1. *$I_u = I_{M(x)}$, $a_y \in D(x)$ and if $I_{M(x)} \neq I_{M(y)}$ then $b_{M(y)} < a_x + 2n$ otherwise $b_{SM(y)} < a_x + 2n$.*
2. *$I_{d(u)} = I_{M(x)}$, $a_y \in D(x)$ and if $I_{M(x)} \neq I_{M(y)}$ then $b_{M(y)} < a_x + 2n$ otherwise $b_{SM(y)} < a_x + 2n$.*
3. *$I_u = I_{M(y)}$, $a_x + 2n \in D(y)$ and if $I_{M(y)} \neq I_{M(d(x))}$ then $b_{M(x)} < a_y$ otherwise $b_{SM(x)} < a_y$.*
4. *$I_{d(u)} = I_{M(y)}$, $a_x + 2n \in D(y)$ and if $I_{M(y)} \neq I_{M(d(x))}$ then $b_{M(x)} < a_y$ otherwise $b_{SM(x)} < a_y$.*

(Proof) Necessity ($\Rightarrow$) obviously holds by Lemma 2. Thus, we only prove sufficiency ($\Leftarrow$).

We show that u is a hinge vertex for x and y if condition 1 of Lemma 3 is satisfied. Assume that there exist intervals I_u, I_x, I_y $(x < y)$ and a dummy interval $I_{d(x)} = [a_x + 2n, b_x + 2n]$ in ECM. It is obvious that $b_x < a_y$ by "$a_y \in D(x)$," i.e., "$b_{SM(x)} < a_y < b_{M(x)}$." Also, clearly, $b_y < a_x + 2n$ by "$b_{M(y)} < a_x + 2n$" or "$b_{SM(y)} < a_x + 2n$." These mean that neither I_x nor $I_{d(x)}$ intersect with I_y. Next, I_u intersects with both I_x and I_y by "$I_u = I_{M(x)}$" and "$a_y \in D(x)$," moreover, any interval less than $I_{M(x)}$ does not intersect with both I_x and I_y. This implies that I_u is the only interval intersecting with both I_x and I_y. At this point we distinguish two cases of "$I_{M(x)} \neq I_{M(y)}$" and "$I_{M(x)} = I_{M(y)}$." In the case of $I_{M(x)} \neq I_{M(y)}$, there exists no interval intersecting with both I_y and $I_{d(x)}$ if $b_{M(y)} < a_x + 2n$ (see Fig. 5-(a)). Moreover, in the case of $I_{M(x)} = I_{M(y)}$, there exists no interval (except I_u or $I_{d(u)}$) intersecting with both I_y and $I_{d(x)}$ if $b_{SM(y)} < a_x + 2n$ (see Fig. 5-(b)). These imply that condition 1 of Corollary 2 holds if condition 1 of Lemma 3 is satisfied, that is, vertex u is a hinge vertex for x and y.

Similarly, conditions 2, 3 and 4 of Corollary 2 hold when conditions 2, 3 and 4 of Lemma 3 are satisfied, respectively. Therefore, u is a hinge vertex for x and y when at least one of the conditions 1, 2, 3 and 4 of Lemma 3 holds. □

An example where vertex 6 is recognized as a hinge vertex in circular-arc graph G illustrated in Fig. 2 by applying condition 1 of Lemma 3 is shown as follows. Assume that $I_x = I_1$, then $I_u = I_{M(x)} = I_{M(1)} = I_6$ and $D(x) = D(1) = \{8, 9, 10, 11, 12\}$. Here, we search some y satisfying that $a_y \in D(x)$ and $x < y$, then, we can find such y's that $y = 5$, 8 and 10. Note that y's satisfying that $a_y \in D(x) = \{8, 9, 10, 11, 12\}$ can be obtained by accessing $ra(k)$ for all $k \in D(x)$ immediately, for example, $ra(9) = 5$, $ra(10) = 8$ and $ra(12) = 10$. Finally, we check whether the following statement holds: if $I_{M(x)} \neq I_{M(y)}$ then $b_{M(y)} < a_x + 2n$ otherwise $b_{SM(y)} < a_x + 2n$ for all pairs of $x = 1$ and $y = 5, 8, 10$. For the pair of $x = 1$ and $y = 5$, vertex $u = M(x) = M(1) = 6$ is recognized as a hinge vertex since "$I_{M(x)} \neq I_{M(y)}$ and $b_{M(y)} = 13 < a_x + 2n = 23$" holds. Similarly, for the pair of $x = 1$ and $y = 8$, a vertex $u = 6$ is recognized as a hinge vertex since "$I_{M(x)} \neq I_{M(y)}$ and $b_{M(y)} = 19 < a_x + 2n = 23$" holds. However, for the pair of $x = 1$ and $y = 10$, "$I_{M(x)} \neq I_{M(y)}$ and $b_{M(y)} < a_x + 2n = 23$" is not satisfied by $b_{M(y)} = 26$. Hence, vertex 6 is not a hinge vertex for $x = 1$ and $y = 10$. In this way, all hinge vertices of G can be found. However, we must check whether at least one of the four conditions of Lemma 3 is satisfied or not for all pairs of x and y $(x < y)$. This takes $O(n^3)$ time in the worst case. Thus, we shall propose a more efficient procedure to find all hinge vertices. The following lemmas are useful for this purpose.

Lemma 4 *Let CM be a circular-arc model, and let $G = (V, E)$ be a circular-arc graph with $x_1, x_2 \in V$ corresponding to CM. Also, let ECM be an extended circular-arc model constructed from CM. Then, $M(x_1) \leq M(x_2)$ and $SM(x_1) \leq SM(x_2)$ for two intervals $I_{x_1} < I_{x_2}$ in ECM.*

(Proof) Assume, to the contrary, that $M(x_1) > M(x_2)$, i.e., $b_{M(x_1)} > b_{M(x_2)}$ for two intervals $I_{x_1} < I_{x_2}$. Then, there exists an interval $I_{M(x_1)}$ that are larger than $I_{M(x_2)}$ and intersects with interval I_{x_2}. This contradicts the fact that $I_{M(x_2)}$ is the largest interval intersecting with I_{x_2}. Similarly, assume, to the contrary, that $SM(x_1) > SM(x_2)$, i.e., $b_{SM(x_1)} > b_{SM(x_2)}$ for two intervals $I_{x_1} < I_{x_2}$. Then, there exist two intervals $I_{SM(x_1)}$ and $I_{M(x_2)}$ that are larger than $I_{SM(x_2)}$ and intersect with interval I_{x_2}. This contradicts the fact that $I_{SM(x_2)}$ is the second largest interval intersecting with I_{x_2}. Thus, $M(x_1) \leq M(x_2)$ and $SM(x_1) \leq SM(x_2)$ for two intervals $I_{x_1} < I_{x_2}$ in ECM. □

Lemma 5 *Let CM be a circular-arc model, and let $G = (V, E)$ be a circular-arc graph with $x_1, x_2 (x_1 < x_2) \in V$ corresponding to CM. Also, let ECM be an extended circular-arc model constructed from CM. Then, $D(x_1) \supseteq D(x_2)$ if $M(x_1) = M(x_2)$ for two intervals $I_{x_1} < I_{x_2}$ in ECM.*

(Proof) $D(x_1) = \{k \mid b_{SM(x_1)} < k < b_{M(x_1)}, k \in \mathcal{N}\}$ and $D(x_2) = \{k \mid b_{SM(x_2)} < k < b_{M(x_2)}, k \in \mathcal{N}\}$ by the definition of $D(i)$. Obviously $b_{M(x_1)} = b_{M(x_2)}$ by the assumption that $M(x_1) = M(x_2)$. Also, $b_{SM(x_1)} \leq b_{SM(x_2)}$ since $I_{x_1} < I_{x_2}$ in ECM by Lemma 4. Hence, $D(x_1) \supseteq D(x_2)$ if $M(x_1) = M(x_2)$ for two intervals $I_{x_1} < I_{x_2}$ in ECM (see Fig. 6). □

The implication of Lemma 5 is as follows. In order to verify whether a vertex u is a hinge vertex, we now test if condition 1 of Lemma 3 holds or not. So far, we must try to find $y's$ satisfying that $a_y \in D(x)$ for all x such that $I_u = I_{M(x)}$. Now, suppose that $M(x_1) =$

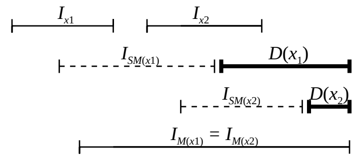


Figure 6: Illustration of Lemma 5.

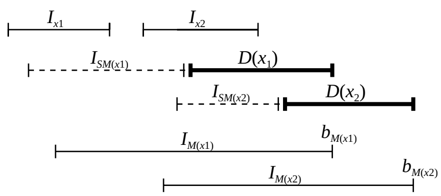


Figure 7: Illustration of Lemma 6.

$M(x_2) = \cdots = M(x_m) = u$ for $x_1 < x_2 < \cdots < x_m$, that is, x_1 is a represent vertex of MV_u. Then, by Lemma 5, $a_y \in D(x_i)$ since $D(x_i) \supseteq D(x_j)$ if $a_y \in D(x_j)$ for $x_i < x_j$. This means that it is sufficient to check whether there exists y satisfying that $a_y \in D(x_1)$ for a represent vertex x_1 of MV_u in the case of $M(x_1) = M(x_2) = \cdots = M(x_m) = u$. Hence, we may only apply condition 1 of Lemma 3 for all represent vertices $x \in RVS$ to check whether $u = M(x)$ is a hinge vertex or not. The number of times for applying condition 1 of Lemma 3 is $\sum_{i \in RVS} |D(i)| = |D_{RVS}|$.

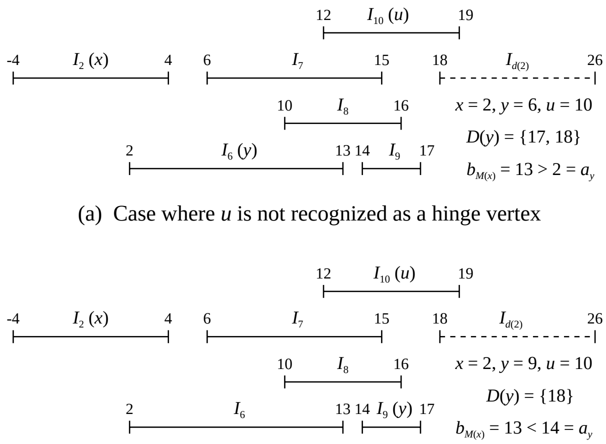
Lemma 6 *Let CM be a circular-arc model, and let $G = (V, E)$ be a circular-arc graph with $x_1, x_2 (x_1 < x_2) \in V$ corresponding to CM. Also, let ECM be an extended circular-arc model constructed from CM. Then, either $M(x_1) = M(x_2)$ or $D(x_1) \cap D(x_2) = \emptyset$ for two intervals $I_{x_1} < I_{x_2}$ in ECM.*

(Proof) Assume, to the contrary, that $M(x_1) \neq M(x_2)$ and $D(x_1) \cap D(x_2) \neq \emptyset$ for two intervals $I_{x_1} < I_{x_2}$ in ECM. We may only consider the case of $M(x_1) < M(x_2)$ since the case of $M(x_1) > M(x_2)$ never happens by Lemma 4. By the definition of $D(i)$, $D(x_1) = \{k \mid b_{SM(x_1)} < k < b_{M(x_1)}, k \in \mathcal{N}\}$ and $D(x_2) = \{k \mid b_{SM(x_2)} < k < b_{M(x_2)}, k \in \mathcal{N}\}$. Now, $b_{M(x_1)} < b_{M(x_2)}$ by $M(x_1) < M(x_2)$, and $b_{SM(x_1)} \leq b_{SM(x_2)}$ by Lemma 4. Also, $b_{SM(x_2)} < b_{M(x_1)}$ when $D(x_1) \cap D(x_2) \neq \emptyset$ (see Fig. 7). Then, there exist two intervals $I_{M(x_1)}$ and $I_{M(x_2)}$ that are larger than $I_{SM(x_2)}$ and intersect with I_{x_2}. This contradicts the fact that $I_{SM(x_2)}$ is the second largest interval intersecting with I_{x_2}. Thus, either $M(x_1) = M(x_2)$ or $D(x_1) \cap D(x_2) = \emptyset$ for two intervals $I_{x_1} < I_{x_2}$ in ECM. $\Box$

The implication of Lemma 6 is as follows. $D(x_1) \cap D(x_2) = \emptyset$ since $M(x_1) \neq M(x_2)$ for two vertices $x_1, x_2 (x_1 < x_2) \in RVS$. This means that all elements of $D(i), i \in RVS$ are distinct, that is, $\sum_{i \in RVS} |D(i)| = |D_{RVS}| \leq 2n$. Hence, the number of times for applying each of conditions 1, 2, 3 and 4 of Lemma 3 to find all hinge vertices is $O(n)$.

So far, we presented a necessary and sufficient condition for recognizing a hinge vertex of a circular-arc graph G in Lemma 3. We further observe that complexity is improved by applying Lemmas 5 and 6. The number of times for applying the necessary and sufficient condition of Lemma 3 is $O(n)$.

However, we must be more careful when we implement an algorithm employing the necessary and sufficient condition of Lemma 3 for finding all hinge vertices of G. We show an example that a hinge vertex is not recognized correctly. Suppose that we check whether vertex 10 is a hinge vertex or not by applying condition 3 of Lemma 3 in an example shown in Table 1. We now choose $y = 6$ by $RVS = \{1, 3, 6, 10\}$, then $u = M(y) = M(6) = 10$. Next, we search x satisfying that $a_x + 2n \in D(y) = D(6) = \{17, 18\}$ and $n = 11$, then we can find $x = 2$ satisfying $a_2 + 22 = 18$. Here, such x's satisfying that $a_x + 2n \in D(y) = \{17, 18\}$ can be obtained by accessing $la(k)$ for all $k \in D(y)$ immediately, that is $x = la(18) = 2$. Now, $M(y) \neq M(d(x))$ by $M(y) = 10$ and $M(d(x)) = 14$. Hence, if $b_{M(x)} < a_y$ for $x = 2$ and $y = 6$, $u = 10$ is recognized as a hinge vertex of G by condition 3 of Lemma 3. At this point, by $b_{M(2)} = 13 > 2 = a_6$, condition 3 of Lemma 3 is not satisfied, then vertex 10 is not recognized as a hinge vertex of G. However, vertex 10 is in fact a hinge vertex for vertices 2 and 9 in G. We explain the reason by using Fig. 8, why this contradiction is occurred. At the first step, we choose $y = 6$ from $RVS = \{1, 3, 6, 10\}$ and found $x = 2$ satisfying that $a_x + 2n = 18 \in D(y) = D(6) = \{17, 18\}$. However, vertex u is not recognized as a hinge vertex since $b_{M(2)} < a_y$ is not satisfied (see Fig. 8-(a)). Here, we pay attention that y satisfying $a_2 + 2n = 18 \in D(y)$ is not only 6, but 7, 8 and 9 also satisfying $a_2 + 2n = 18 \in D(y)$ ($D(7) = D(8) = D(9) = \{18\}$). If there exists y satisfying $b_{M(2)} < a_y$ among y's ($y = 6, 7, 8, 9$) satisfying that $a_2 + 2n = 18 \in D(y)$, a vertex u is recognized as a hinge vertex for x and y. Therefore, we must choose m as y such that a_m is the maximum value of $\{a_i \mid 18 \in D(i)\}$. For an example shown in Fig. 8-(b), $i = 9$ is chosen as y by $a_6 = 2$, $a_7 = 6$, $a_8 = 10$ and $a_9 = 14$ for $18 \in D(i)$ (that is, $i = 6, 7, 8$ and 9). Then, $I_u = I_{10} = I_{M(y)}$, $a_x + 2n = 18 \in D(y) = \{18\}$ and $b_{M(x)} = 13 < 14 = a_y$ for $y = 9$ and $x = 2$, hence, vertex 10 is correctly recognized as a hinge vertex for vertices 2 and 9. We describe how to choose y again. We must not directly choose some element j of RVS as y. We must choose j as y such that a_j is the maximum value of all a_i satisfying $k \in D(i)$ for $k \in D_{RVS}$. Note that such y can be obtained by accessing $p(k)$ immediately, for example, $p(18) = 9$.



(a) Case where *u* is not recognized as a hinge vertex

(b) Case where *u* is correctly recognized as a hinge vertex

Figure 8: Example for how to find a hinge vertex u.

Hence, the necessary and sufficient condition for a hinge vertex described in Lemma 7 is replaced as follows.

Lemma 7 *Let CM be a circular-arc model, and let $G = (V, E)$ be a circular-arc graph with $x, y (x < y) \in V$ corresponding to CM. Also, let ECM be an extended circular-arc model constructed from CM. Vertex u is a hinge vertex for x and y of G if and only if one of at least one of the following four conditions holds for $k \in D_{RVS}$ in ECM.*

1. *$y = ra(k)$, $x = p(k)$, $u = M(x)$, and if $M(x) \neq M(y)$ then $b_{M(y)} < a_x + 2n$ otherwise $b_{SM(y)} < a_x + 2n$.*
2. *$y = ra(k)$, $x = p(k)$, $d(u) = M(x)$, and if $M(x) \neq M(y)$ then $b_{M(y)} < a_x + 2n$ otherwise $b_{SM(y)} < a_x + 2n$.*
3. *$x = la(k)$, $y = p(k)$, $u = M(y)$, and if $M(y) \neq M(d(x))$ then $b_{M(x)} < a_y$ otherwise $b_{SM(x)} < a_y$.*
4. *$x = la(k)$, $y = p(k)$, $d(u) = M(y)$, and if $M(y) \neq M(d(x))$ then $b_{M(x)} < a_y$ otherwise $b_{SM(x)} < a_y$.*

4 Algorithm

We introduce a parallel algorithm PHV for finding all hinge vertices of a circular-arc graph as follows.

Algorithm PHV

Input: $[a_i, b_i]$, $1 \leq i \leq n$, coordinates of left and right endpoints of circular-arcs A_i in CM.
Output: The set HV of all hinge vertices. Initially, $HV := \emptyset$.

Step 1 (Construction of ECM)

Step 2 (Construction of $M(i)$, $SM(i)$ and $D(i)$)

Step 3 (Construction of $MV(i)$, RVS and D_{RVS})
  Make an array $MV(1..n)$
  /* construction of $MV(i)$ */
  For all i, $1 \leq i \leq n$, **in parallel do**
    $MV(i) := \emptyset$
    If $D(i) \neq \emptyset$ **then**
      $MV(M(i)) := MV(M(i)) \cup \{i\}$
  End parallel
  /* construction of RVS and D_{RVS} */
  $RVS := \{1\}$
  $D_{RVS} := D(1)$
  For all i, $2 \leq i \leq n$, **in parallel do**
    If $(M(i-1) \neq M(i)) \wedge (D(i) \neq \emptyset)$ **then**
      $RVS := RVS \cup \{i\}$
      $D_{RVS} := D_{RVS} \cup D(i)$
  End parallel

Step 4 (Construction of $la(k)$, $ra(k)$)
  $l := \min\{a_i \mid I_i \text{ in } ECM\}$
  $r := \max\{a_i \mid I_i \text{ in } ECM\}$
  $ld := \min\{i \mid i \in D_{RVS}\}$
  $rd := \max\{i \mid i \in D_{RVS}\}$
  Make arrays $la(ld..rd)$ and $ra(ld..rd)$
  For all k, $k \in D_{RVS}$ **in parallel do**
    /* construction of $la(k)$
    If $AUX_5(k) \neq \emptyset$ **then** $la(k) := AUX_5(k)$
    /* construction of $ra(k)$
    If $AUX_6(k) \neq \emptyset$ **then** $ra(k) := AUX_6(k)$
  End parallel

Step 5 (Construction of $p(k)$)
  Make arrays $AUX_7(1..n)$, $AUX_8(1..n)$ and $p(1..2n)$
  For all $v \in RVS$ **in parallel do**
    $l(v) := \min\{MV(v)\}$
    $r(v) := \max\{MV(v)\}$
    /* parallel prefix max computation */
    For all i, $l(v) \leq i \leq r(v)$ **in parallel do**
      $AUX_7(i) := \max\{a_{l(v)}, a_{l(v)+1}, \ldots, a_i\}$
    End parallel
    For all i, $l(v) \leq i \leq r(v) - 1$ **in parallel do**
      $AUX_8(i) := D(i) \setminus D(i+1)$
    End parallel
    $AUX_8(r(v)) := D(v)$
  End parallel
  For all i, $AUX_8(i) \neq \emptyset$, $1 \leq i \leq n$ **in parallel do**
    For all $k \in AUX_8(i)$ **in parallel do**
      $p(k) := AUX_2(AUX_7(k))$
    End parallel
  End parallel

Step 6 (Finding all hinge vertices)
  For all k, $k \in D_{RVS}$ **in parallel do**
    $y := ra(k)$, $x := p(k)$, $u := M(x)$
    If $\{(M(x) \neq M(y)) \wedge (b_{M(y)} < a_x + 2n)\}$
      $\vee\{(M(x) = M(y)) \wedge (b_{SM(y)} < a_x + 2n)\}$
        then $HV := HV \cup \{u\}$

$y := ra(k)$, $x := p(k)$, $d(u) := M(x)$
If $\{(M(x) \neq M(y)) \wedge (b_{M(y)} < a_x + 2n)\}$
$\vee\{(M(x) = M(y)) \wedge (b_{SM(y)} < a_x + 2n)\}$
then $HV := HV \cup \{u\}$
$x = la(k)$, $y = p(k)$, $u := M(y)$
If $\{(M(y) \neq M(d(x))) \wedge (b_{M(x)} < a_y)\}$
$\vee\{(M(y) = M(d(x))) \wedge (b_{SM(y)} < a_y)\}$
then $HV := HV \cup \{u\}$
$x = la(k)$, $y = p(k)$, $d(u) := M(y)$
If $\{(M(y) \neq M(d(x))) \wedge (b_{M(x)} < a_y)\}$
$\vee\{(M(y) = M(d(x))) \wedge b_{SM(y)} < a_y)\}$
then $HV := HV \cup \{u\}$
End parallel

After executing Step 6, HV consists of the set of all hinge vertices of G.

End of Algorithm PHV

We shall describe details of parallel algorithm PHV and analyze the complexity. Parallel algorithm PHV finds all hinge vertices of a circular-arc graph G based on the necessary and sufficient condition for a hinge vertex described in Lemma 7. At first, we assume that the circular-arcs in CM are already sorted with respect to values of endpoints b's. In Step 1, we construct ECM from CM. This step are computed in $O(1)$ time using $O(n)$ processors, which can be implemented in $O(\log n)$ time using $O(n/\log n)$ processors by applying Brent's scheduling principle [13]. In Step 2, we compute $M(i)$, $SM(i)$ and $D(i)$ for i, $1 \leq i \leq n$. This step can be implemented in $O(\log n)$ time with $O(n/\log n)$ processors by applying parallel prefix computation [14]. In Step 3, we construct $MV(i)$, RVS and D_{RVS}. This step can be implemented in $O(\log n)$ time with $O(n/\log n)$ processors by applying Brent's scheduling principle. In Step 4, we construct arrays $la(k)$ and $ra(k)$, $k \in D_{RVS}$. This step can be executed in $O(\log n)$ time with $O(n/\log n)$ processors by applying Brent's scheduling principle. In Step 5, we construct array $p(k)$, $k \in D_{RVS}$. By initializing $AUX_8(i) := D(i) \backslash D(i+1)$, elements of $AUX_8(i)$ for $1 \leq i \leq n$ becomes distinct, and we can obtain $p(k)$ in $O(n)$ work complexity in last 'For' statement. Thus, this step can be implemented in $O(\log n)$ time with $O(n/\log n)$ processors by applying parallel prefix computation. In Step 6 we find all hinge vertices by applying a necessary and sufficient condition of Lemma 7. This step can be executed in $O(\log n)$ time with $O(n/\log n)$ processors. The concurrent reading and writing are not used anywhere. Hence we have the following theorem.

Theorem 2 *Given a circular-arc graph G, Algorithm PHV finds the set of all hinge vertices of G in $O(\log n)$ time using $O(n/\log n)$ processors on EREW PRAM.* □

5 Concluding Remarks

In this paper, we have presented an optimal parallel algorithm for finding all hinge vertices of a circular-arc graph. When the graph is given in the form of a family of n arcs on a circle, our algorithm runs in $O(\log n)$ time with $O(n/\log n)$ processors in EREW PRAM model. In the future, we are interested in solving this problem on some special graphs such as circle graphs, circle trapezoid graphs and so on.

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