New aspects of the parameter space and Feynman integral calculations

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Abstract

We review the new parameter transformation we presented in the previous papers ten years ago. We show several structures of this transformation. Further using this parameter transformation we calculate the vertex function in Feynman integrals to obtain the anomalous magnetic moment of an on-shell quark in the scheme of the perturbative QCD theory and the dimensional regularization. And we perform the same calculation by using the beautiful Mellin Barnes representation method and compare the result of this calculation to the one by using our new transformation method.

Key Words : Feynman integrals, quantum chromodynamics, hypergeometric function, Manifold

1 Introduction

Recently many important results have been obtained in high energy physics including the detection of Higgs boson owing to the operation of LHC and LEP at CERN.

When we try to compare the theory, especially QCD, with such data and predict the new phenomena, we have to calculate Feynman integrals finally. Ten years ago we proposed the new parameter transformation and the method to calculate the Feynman propagators. Many particle physicists have used this method partially to calculate Feynman propagators. After that a lot of methods were discovered to calculate the propagators within the dimensional regularization theory, that is, (i) the negative dimension method[1][2] (ii) the epsilon expansion technique[3] (iii) the method applying the Mellin Barnes representation[4][5] (iv) the method using hyper geometric functions[6][7], and so on.

Until now we had two ways to calculate Feynmam integrals simply, the so called Feynman parametrization method and Schwinger's parametrization method. Furthermore we found the new parameter transformation and its integral method in the previous paper[8][9]. This new parameter transformation was what we solve Schwinger's parametrization $x = \alpha(1-\beta), y = \alpha\cdot\beta$ in reverse and generalize the solutions $\alpha = x + y, \beta = \frac{y}{x+y}$. We could show that the results of the calculations concerning Feynman integrals by using our new parametrization and its integral method are equal completely to the results calculated by using the usual Feynman parametrization technique on the loop level in the scheme of the dimensional regularization.

In sec.**2** we review our parameter transformation space coordinate and explain its several

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structures and the new integral method in the parameter space. In sec.**3** we calculate the simplest Feynman amplitude concerning the vertex function to calculate the anomalous magnetic moment of an on-shell quark by using our method of parametrization. In sec.**4** we try to calculate the same vertex function as in sec.**3** exploiting the beautiful Mellin Barnes representation method, and compare the result with the one in sec.**3**.

2 Our parameter transformation

We consider the parameter transformation from $(t_1, t_2, \cdots, t_n)$ to $(x_0, x_1, \cdots, x_{n-1})$,

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} t_i \\ \frac{t_1}{\sum_{i=1}^{n} t_i} \\ \frac{t_2}{\sum_{i=1}^{n} t_i} \\ \vdots \\ \frac{t_{n-1}}{\sum_{i=1}^{n} t_i} \end{pmatrix} = \begin{pmatrix} x_0 \\ \frac{t_1}{x_0} \\ \frac{t_2}{x_0} \\ \vdots \\ \frac{t_{n-1}}{x_0} \end{pmatrix} \qquad (1)$$

where $x_0 = \sum_{i=1}^{n} t_i$. We should remember that x_0 is linear on t_i , $x_i (i \neq 0)$is non-linear on $x_0 = \sum_{i=1}^{n} t_i$. But there exists the following constraint:

$$x_0 = \sum_{i=1}^{n} t_i, \quad \text{and} \quad \sum_{i=1}^{n} x_i = 1, \qquad (2)$$

where $x_n = t_n / \sum_{i=1}^{n} t_i$ and we have to pay attention to the fact that this x_n is not included in the set V. Here we define the correspondence $U \longrightarrow V$, where U is the set $U = \{t_1, t_2, \cdots, t_n\}$ and V is the set $V = \{x_0, x_1, \cdots, x_{n-1}\}$. We can summarize that the correspondence $U \longrightarrow V$ has the next properties.

1. The set V is projective space.
2. The correspondence $U \longrightarrow V$ is injective under the condition $x_0 = \sum_{i=1}^{n} t_i = c$(constant).
3. $U \cap V = \phi$ (empty set)
4. An element constructed out of $t_i (i = 1, 2, \cdots, n)$ exists in ϕ. In (1) it fits in with $x_n = t_n \Big/ \sum_{i=1}^{n} t_i$.
5. The transformation is a statistical one.

Therefore the set V is the so-called Hausdorff space mathematically and has manifold features when there is the next condition $x_0 = \sum_{i=1}^{n} t_i =$ const. We can calculate the Jacobian of this transformation:

$$\text{abs}\Big[\frac{\partial(t_1, t_2, \cdots, t_n)}{\partial(x_0, x_1, \cdots, x_{n-1})}\Big] = x_0^{n-1} \qquad (3)$$

We proved the equation(3) precisely using the properties of a determinant.See the equation(36) of Appendix A. We can consider the symmetric transformation $\Big(x_i\Big) = \Big(t_i / \sum_{i=1}^{n} t_i\Big)$as well, without including the element $x_0 = \sum_{i=1}^{n} t_i$, but the Jacobian becomes ∞, so that we can't well-define the integration using this symmetric transformation. Refer to Appendix B. The partial differential equations hold true between x_i and t_i,

$$\begin{cases} \frac{\partial^2 x_0}{\partial t_i \partial t_j} = 0 \\ \frac{\partial^2 x_k}{\partial t_i \partial t_j} = \frac{2}{x_0{}^2} x_k - \frac{\delta_{ik}}{x_0{}^2} - \frac{\delta_{jk}}{x_0{}^2} \ \ (k \neq 0) \end{cases} \qquad (4)$$

where δ_{ik} is Kronecker's delta. The character of these equations is that on the one hand the equation on x_0 is linear on t_i and t_j, and on the other hand the equations on $x_i (i \neq 0)$ are hyperbolic function-like if x_0 is constant, that is, on the flat sheet $x_0 = c$ in the parameter space.

Using this transformation we can define the integral as follows:

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty f(t_1, t_2, t_3, \ldots, t_n)\, dt_1\, dt_2 \ldots\, dt_n = \qquad (5)$$

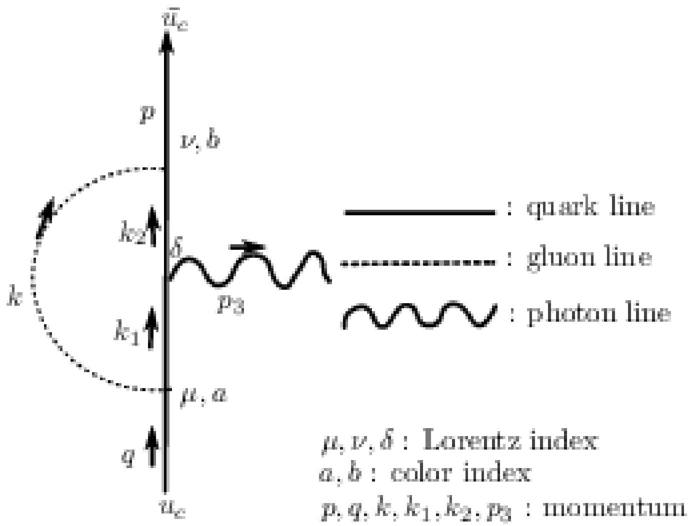

Fig.1 The anomalous magnetic moment diagram in QCD

$$= \int_0^\infty \text{abs}\Big[\frac{\partial(t_1, t_2, \ldots, t_n)}{\partial(x_0, x_1, \ldots, x_{n-1})}\Big] dx_0 \int_0^1 \int_0^1 \ldots \int_0^1 f(x_0, x_1, \ldots, x_n) \times$$
$$\times \delta(1 - \sum_{i=1}^{n} x_i)\, dx_1 \ldots dx_n$$

The number of the element dx_i is different from the number of the element dt_i,but because there exists δ-function $\delta(1 - \sum_{i=1}^{n} x_i)$, after the surface integral on x_n both numbers of element are equal. It is illustrated graphically in [5][6] using Maple software that the integral domain of the variable x_0 is from 0 to positive infinity and the domains of the variables $x_i (i = 1, 2, \cdots)$are from 0 to 1 . The δ-function contained within the right hand side in (5) comes from the constraint $\sum_{i=1}^{n} x_i = 1$.

3 The calculation of the simplest vertex function in QCD by our parametrization technique

We consider the simplest vertex function in QCD. The simplest Feynman amplitude is

$$I(p, q) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(p-k)^2 - m_c^2 + i\eta][(q-k)^2 - m_c^2 + i\eta][k^2 + i\eta]}, \tag{6}$$

where p , q are the momenta of the outer lines with the mass m_c (a quark mass) and k is the gluon momentum of the inner line.
The Feynman graph is illustrated in Fig.1. First of all we calculate the Feynman integral (6) by using our parametrization, its integral method and the dimensional regularization. Actually there exists the complex numerator N including γ matrix and the spinor calculation, but fundamentally it is known that if the equation (6) can be calculated,the integration including the numerator N can be done, too. See the reference [8][9] about the specific calculation of N. We Wick-rotate the Feynman integral (6) from Minkowski momentum space p, q, k to Euclidean momentum space p', q', k' and introduce the exponential parametrization. Now we eliminate the infinitesimal $i\eta$ because the denominator doesn't have any poles.

$$I = i\frac{1}{(2\pi)^4} \int d^4k' \frac{-1}{[(p'-k')^2 + m_c^2][(q'-k')^2 + m_c^2][k'^2]} = \tag{7}$$
$$= \quad -i \int \frac{1}{(2\pi)^4} d^4k' \prod_{i=1}^{3} \int_0^\infty dt_i \exp\Big[-((p'-k')^2 + m_c^2)t_1 - ((q'-k')^2 + m_c^2)t_2 - k'^2 t_3\Big]$$

Now we introduce on-shell conditions $p'^2 = -m_c^2$, $q'^2 = -m_c^2$ for simplicity and have the following variable transformation

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^3 t_i \\ \frac{t_1}{x_0} \\ \frac{t_2}{x_0} \end{pmatrix}, \tag{8}$$

where we put $x_0 = \sum_{i=1}^3 t_i$.
The Jacobian of this variable transformation becomes as follows:

$$J = \mathrm{abs}\Big[\frac{\partial(t_1, t_2, t_3)}{\partial(x_0, x_1, x_2)}\Big] = x_0^2. \tag{9}$$

Using (5) and (9) we can change the integral variables in (7) from t_i to x_i.

$$\begin{aligned} I &= -i \int \frac{1}{(2\pi)^4} d^4k' \prod_{i=1}^3 \int_0^\infty dt_i \\ &\quad \times \exp\Big[-\textstyle\sum_{i=1}^3 t_i (k'^2 - 2p'k't_1/\sum_{i=1}^3 t_i - 2q'k't_2/\sum_{i=1}^3 t_i)\Big] \\ &= -i \int \frac{1}{(2\pi)^4} d^4k' \int_0^\infty \mathrm{abs}\left[\frac{\partial(t_1, t_2, t_3)}{\partial(x_0, x_1, x_2)}\right] dx_0 \int_0^1 dx_1 \int_0^1 dx_2 \\ &\quad \times \int_0^1 dx_3 \delta(1 - \textstyle\sum_{i=1}^3 x_i) \exp[-x_0(k'^2 - 2p'k'x_1 - 2q'k'x_2)] \end{aligned} \tag{10}$$

If we integrate on x_3 , we get the following equation.

$$\begin{aligned} I &= -i \int \frac{1}{(2\pi)^4} d^4k' \int_0^\infty dx_0 x_0^2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \times \\ &\quad \times \exp[-x_0((k' - p'x_1 - q'x_2)^2 - 2p'q'x_1x_2 - p'^2x_1^2 - q'^2x_2^2)] \end{aligned} \tag{11}$$

Performing the 2ω dimensional momentum extension, introducing the 2ω dimensional polar coordinate and integrating the angular parts, the integral (11) becomes as follows:

$$\begin{aligned} I &= -i \frac{\pi^\omega}{(2\pi)^{2\omega}\Gamma(\omega)} \int_0^\infty (\tilde{k}^2)^{\omega-1} d(\tilde{k}^2) \int_0^\infty dx_0 x_0^2 \int_0^1 dx_1 \\ &\quad \times \int_0^{1-x_1} dx_2 \exp[-x_0\tilde{k}^2] exp[-m_c^2(x_1 + x_2)^2 x_0] \end{aligned} \tag{12}$$

where we used the relation $p'q' = -m_c^2$, and shifted the momentum from k' to $\tilde{k}' = k' - p'x_1 - q'x_2$ and further we rewrote $\tilde{k}'$ (four dimensional momentum) in $\tilde{k}$ (2ω dimensional momentum). See Appendix B in [8] or refer to [9].
Putting $y = x_0\tilde{k}^2$, we have $\tilde{k}^2 = y/x_0$ and $d(\tilde{k}^2) = dy/x_0$, so that the integral is:

$$\begin{aligned} I &= -i \frac{1}{(4\pi)^\omega \Gamma(\omega)} \int_0^\infty y^{\omega-1} \exp[-y]\, dy \int_0^\infty x_0^{2-\omega}\, dx_0 \int_0^1 dx_1 \\ &\quad \times \int_0^{1-x_1} dx_2 \exp[-m_c^2(x_1 + x_2)^2 x_0] \\ &= -i \frac{\Gamma(\omega)}{(4\pi)^\omega \Gamma(\omega)} \int_0^\infty dx_0 x_0^{2-\omega} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \exp[-m_c^2(x_1 + x_2)^2 x_0] \end{aligned} \tag{13}$$

Further performing the following variable transform $z = m_c^2(x_1 + x_2)^2 x_0$, $dx_0 = dz/m_c^2(x_1 + x_2)^2$, we have:

$$\begin{aligned} I &= \frac{-i}{(4\pi)^\omega} \int_0^\infty z^{2-\omega} \exp[-z]\, dz \int_0^1 dx_1 \int_0^{1-x_1} dx_2 (m_c^2(x_1 + x_2)^2))^{\omega-3} \\ &= \frac{-i}{(4\pi)^\omega} \Gamma(3 - \omega)(m_c^2)^{\omega-3} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 (x_1 + x_2)^{2\omega-6} \end{aligned} \tag{14}$$

The final integral is very trivial and we can obtain the result as follows:

$$I = \frac{-i}{(4\pi)^{\omega}} \frac{\Gamma(3-\omega)}{m_c^2} (m_c^2)^{\omega-2} \frac{1}{2\omega - 4}. \tag{15}$$

Finally

$$I = \frac{i}{2(4\pi)^{\omega}} \frac{\Gamma(2-\omega)}{m_c^2} (m_c^2)^{\omega-2}. \tag{16}$$

.

4 The calculation by using Mellin Barnes representation

We try to calculate the vertex integral with two massive denominators and the same masses m_c. In this case we adopt the method applying the Mellin Barnes representation discovered by A.I.Davydychev and others. From the equation (6)

$$I_3(p,q) = \int \frac{dk}{(2\pi)^4} \frac{1}{[(p-k)^2 - m_c^2 + i\eta][(q-k)^2 - m_c^2 + i\eta][k^2 + i\eta]} \tag{17}$$

We drop the infinitesimal quantity $i\eta$ for convenience as pseudo Euclidean space. At first we change one of the denominator factors as follows :

$$\begin{aligned} \frac{1}{\left[(p-k)^2 - m_c{}^2\right]} &= \frac{1}{(p-k)^2\left[1 - \frac{m_c{}^2}{(p-k)^2}\right]} \\ &= \frac{1}{(p-k)^2} \sum_{j_1=0}^{\infty} \frac{(1)_{j_1}}{j_1!} \Big(\frac{m_c{}^2}{(p-k)^2}\Big)^{j_1} \\ &= \frac{1}{(p-k)^2} {}_1F_0\left(1; \frac{m_c{}^2}{(p-k)^2}\right), \end{aligned} \tag{18}$$

where ${}_mF_n(\alpha_1, \alpha_2, \ldots, \alpha_m; \beta_1, \beta_2, \ldots, \beta_n; z)$ is hyper-geometric function and

$$(\alpha)_n = (\alpha)(\alpha+1)(\alpha+2)\cdots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha) \tag{19}$$

is, so called, Pochhammer's symbol. Now we consider the Mellin Barnes representation of ${}_1F_0$ as follows:

$${}_1F_0(\alpha; z) = \frac{1}{(2\pi i)\Gamma(\alpha)} \int_{-i\infty}^{i\infty} \Gamma(s+\alpha)\Gamma(-s)(-z)^s\, ds \tag{20}$$

Then using Mellin Barnes formula (20), we can write down the equation (18) as follows

$$\begin{aligned} I_p = \frac{1}{\left((p-k)^2 - m_c{}^2\right)} &= \frac{1}{(p-k)^2} {}_1F_0\left(1; \frac{m_c{}^2}{(p-k)^2}\right) \\ = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} &\Gamma(v+1)\Gamma(-v)(-m_c{}^2)^v \frac{1}{\left((p-k)^2\right)^{v+1}}\, dv. \end{aligned} \tag{21}$$

Similarly

$$\begin{aligned} I_q = \frac{1}{\left((q-k)^2 - m_c{}^2\right)} &= \frac{1}{(q-k)^2} {}_1F_0\left(1; \frac{m_c{}^2}{(q-k)^2}\right) \\ = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} &\Gamma(w+1)\Gamma(-w)(-m_c{}^2)^w \frac{1}{\left((q-k)^2\right)^{w+1}}\, dw. \end{aligned} \tag{22}$$

Substituting the equations (21) and (22) to the equation (17) we obtain the following equation.

$$
\begin{aligned}
& I_3(p,q) = \int \frac{d^n k}{[(p-k)^2 - m_c{}^2][(q-k)^2 - m_c{}^2][k^2]} \\
= & \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} dv\, dw \Gamma(v+1)\Gamma(w+1)\Gamma(-v)\Gamma(-w)(-m_c{}^2)^v \\
& \times (-m_c{}^2)^w \int \frac{d^n k}{(k^2)[(p-k)^2]^{v+1}[(q-k)^2]^{w+1}} \\
= & \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} dv\, dw \Gamma(v+1)\Gamma(w+1)\Gamma(-v)\Gamma(-w)(-m_c{}^2)^v \\
& \times (-m_c{}^2)^w J_3^{(0)}(1, v+1, w+1; 0)
\end{aligned} \tag{23}
$$

where $J_3^{(0)}(1, v+1, w+1)$ represents a massless propagator. Its solution is found in the paper precisely.[11] And the dimension of momentum is extended from 4-dimension to $n = 2\omega$-dimension. Further for simplicity the normalization factor $(2\pi)^{2\omega}$ is abbreviated. The result is

$$
\begin{aligned}
& J_3^{(0)}(1, v+1, w+1; 0) = (i)^{1-n} (\pi)^{\frac{n}{2}} (k^2)^{\frac{n}{2}-3-v-w} \Big[\Gamma(v+1)\Gamma(w+1)\Gamma(n-3-v-w)\Big]^{-1} \\
& \times \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} ds\, dt \Big(\frac{p^2}{k^2}\Big)^s \Big(\frac{q^2}{k^2}\Big)^t \Gamma(-s)\Gamma(-t)\Gamma(\frac{n}{2} - 2 - v - s) \\
& \times \Gamma(\frac{n}{2} - 2 - w - t)\Gamma(1+s+t)\Gamma(3 - \frac{n}{2} + v + w + s + t)
\end{aligned} \tag{24}
$$

where $k^2 = (p-q)^2$ is different from k in the equation (23) and corresponds to Mandelstam variable s. Next substituting the equation (24) into (23) we can obtain the integral

$$
\begin{aligned}
& I_{(3)} = \pi^{\frac{n}{2}} i^{1-n} (k^2)^{\frac{n}{2}-3} \frac{1}{\Big(2\pi i\Big)^4} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} dv\, dw\, ds\, dt \Big(-\frac{m_c{}^2}{k^2}\Big)^v \\
& \times \Big(-\frac{m_c{}^2}{k^2}\Big)^w \Big(\frac{p^2}{k^2}\Big)^s \Big(\frac{q^2}{k^2}\Big)^t \Gamma(-v)\Gamma(-w)\Gamma(-s)\Gamma(-t)\Gamma\Big(\frac{n}{2} - 2 - v - s\Big) \\
& \times \frac{\Gamma\Big(\frac{n}{2} - 2 - w - t\Big)\Gamma\Big(1+s+t\Big)\Gamma\Big(3 - \frac{n}{2} + v + w + s + t\Big)}{\Gamma\Big(n - 3 - v - w\Big)}.
\end{aligned} \tag{25}
$$

Changing the variable w into $w = \frac{n}{2} - 3 - s - t - u - v$, we can transform the integral variables. That is

$$
\begin{aligned}
& I_{(3)} = \pi^{\frac{n}{2}} i^{1-n} \Big(-m_c{}^2\Big)^{\frac{n}{2}-3} \frac{1}{\Big(2\pi i\Big)^4} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} dv\, du\, ds\, dt \Big(-\frac{k^2}{m_c{}^2}\Big)^u \\
& \times \Big(-\frac{p^2}{m_c{}^2}\Big)^s \Big(-\frac{q^2}{m_c{}^2}\Big)^t \Gamma(-s)\Gamma(-t)\Gamma(-u)\Gamma(-v)\Gamma\Big(1+s+t\Big) \\
& \times \frac{\Gamma\Big(\frac{n}{2} - 2 - s - v\Big)\Gamma\Big(1+s+u+v\Big)\Gamma\Big(3 - \frac{n}{2} + s + t + u + v\Big)}{\Gamma\Big(\frac{n}{2} + s + t + u\Big)}.
\end{aligned} \tag{26}
$$

Now remembering Barnes Lemma[10]

$$
\begin{aligned}
& \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)\, ds \\
= & \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)},
\end{aligned} \tag{27}
$$

we can perform the integration of the equation (26) concerning the variable v. The result is

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma\Big(3-\frac{n}{2}+s+t+u+v\Big)\Gamma\Big(1+s+u+v\Big)\Gamma\Big(-v\Big)\Gamma\Big(\frac{n}{2}-2-v\Big)\,dv$$

$$= \frac{\Gamma\Big(3-\frac{n}{2}+s+t+u\Big)\Gamma\Big(1+t+u\Big)\Gamma\Big(1+s+u\Big)\Gamma\Big(\frac{n}{2}-1+u\Big)}{\Gamma\Big(2+2u+s+t\Big)}. \tag{28}$$

Putting the equation (28) into the equation (26), we have

$$\begin{aligned} I_{(3)} &= \pi^{\frac{n}{2}}i^{1-n}\Big(-m_c{}^2\Big)^{\frac{n}{2}-3}\frac{1}{\Big(2\pi i\Big)^3}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}du\,ds\,dt\Big(-\frac{k^2}{m_c{}^2}\Big)^u \\ &\times\Big(-\frac{p^2}{m_c{}^2}\Big)^s\Big(-\frac{q^2}{m_c{}^2}\Big)^t\Gamma(-s)\Gamma(-t)\Gamma(-u) \\ &\times\frac{\Gamma\Big(1+s+t\Big)\Gamma\Big(3-\frac{n}{2}+s+t+u\Big)}{\Gamma\Big(\frac{n}{2}+s+t+u\Big)} \\ &\times\frac{\Gamma\Big(1+s+u\Big)\Gamma\Big(1+t+u\Big)\Gamma\Big(\frac{n}{2}-1+u\Big)}{\Gamma\Big(2+s+t+2u\Big)}. \end{aligned} \tag{29}$$

We can calculate the integration of the equation (29) by making use of the residue theorem on complex variable calculus. The contour of the integration runs on the straight line of the imaginary axis of the complex planes(u, s, t), respectively, from $-i\infty$ to $i\infty$, and draws a large semicircle of the half right plane turning clockwise around the origin in the regions $|\arg(u, s, t)| \leq \pi/2$. Considering the relation

$$\Gamma(-s) = \frac{\Gamma(j+1-s)}{(j-s)(j-1-s)\cdots(1-s)(-s)} \tag{30}$$

and the similar relations of $\Gamma(-t)$ and $\Gamma(-u)$, we can decide the positions of poles and calculate the residue integrations of the equation (29) easily. That is, the result becomes as

$$\begin{aligned} I_{(3)} &= \pi^{\frac{n}{2}}i^{1-n}\Big(-m_c{}^2\Big)^{\frac{n}{2}-3}\sum_{j_1,j_2,j_3=0}^{\infty}\left(-\frac{k^2}{m_c{}^2}\right)^{j_1}\left(-\frac{p^2}{m_c{}^2}\right)^{j_2}\left(-\frac{q^2}{m_c{}^2}\right)^{j_3} \\ &\times\frac{(-1)^{j_1+j_2+j_3}}{j_1!j_2!j_3!}\frac{\Gamma\Big(3-\frac{n}{2}+j_1+j_2+j_3\Big)\Gamma\Big(1+j_1+j_2\Big)}{\Gamma\Big(\frac{n}{2}+j_1+j_2+j_3\Big)} \\ &\times\frac{\Gamma\Big(1+j_1+j_3\Big)\Gamma\Big(1+j_2+j_3\Big)\Gamma\Big(\frac{n}{2}-1+j_1\Big)}{\Gamma\Big(2+2j_1+j_2+j_3\Big)}. \end{aligned} \tag{31}$$

Finally we can write the result in terms of a generalized hyper-geometric function of three variables with the help of the formula (19) as

$$\begin{aligned} I_{(3)} &= \pi^{\frac{n}{2}}i^{1-n}\Big(-m_c{}^2\Big)^{\frac{n}{2}-3}\frac{\Gamma(3-\frac{n}{2})\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2})}\sum_{j_1,j_2,j_3=0}^{\infty}\frac{1}{j_1!j_2!j_3!} \\ &\times\left(\frac{k^2}{m_c{}^2}\right)^{j_1}\left(\frac{p^2}{m_c{}^2}\right)^{j_2}\left(\frac{q^2}{m_c{}^2}\right)^{j_3}\frac{\Big(3-\frac{n}{2}\Big)_{j_1+j_2+j_3}\Big(1\Big)_{j_1+j_2}\Big(1\Big)_{j_1+j_3}\Big(1\Big)_{j_2+j_3}\Big(\frac{n}{2}-1\Big)_{j_1}}{\Big(\frac{n}{2}\Big)_{j_1+j_2+j_3}\Big(2\Big)_{2j_1+j_2+j_3}}. \end{aligned} \tag{32}$$

Therefore

$$I_{(3)} = \pi^{\frac{n}{2}} i^{1-n} \Big(-m_c{}^2 \Big)^{\frac{n}{2}-3} \frac{\Gamma(3-\frac{n}{2})\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2})}$$
$$\times \Phi_3 \Bigg[3-\frac{n}{2}, 1, 1, 1, \frac{n}{2}-1; \frac{n}{2}, 2; \frac{p^2}{m_c{}^2}, \frac{q^2}{m_c{}^2}, \frac{k^2}{m_c{}^2} \Bigg], \tag{33}$$

where

$$\Phi_3 \Bigg[A, B, C, D, E; F, G; x, y, z \Bigg] \tag{34}$$
$$= \sum_{a,b,c=0}^{\infty} \frac{1}{a!b!c!} x^a y^b z^c \frac{(A)_{a+b+c}(B)_{a+b}(C)_{a+c}(D)_{b+c}(E)_c}{(F)_{a+b+c}(G)_{a+b+2c}}.$$

When $k^2 = (p-q)^2 = 0$, we have $j_1 = 0$, and

$$I_{(3)} = \pi^{\frac{n}{2}} i^{1-n} \Big(-m_c{}^2 \Big)^{\frac{n}{2}-3} \frac{\Gamma(3-\frac{n}{2})\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2})} \tag{35}$$
$$\times \sum_{j_2,j_3=0}^{\infty} \frac{1}{j_2!j_3!} \Big(\frac{p^2}{m_c{}^2} \Big)^{j_2} \Big(\frac{q^2}{m_c{}^2} \Big)^{j_3} \frac{\Big(3-\frac{n}{2}\Big)_{j_2+j_3} \Big(1\Big)_{j_2} \Big(1\Big)_{j_3} \Big(1\Big)_{j_2+j_3}}{\Big(\frac{n}{2}\Big)_{j_2+j_3} \Big(2\Big)_{j_2+j_3}}.$$

It is not clear whether the result is equal to (15) and (16) except for the coefficient factors, because the analytic forms or asymptotic behaviors of hyper-geometric function Φ_3 are unknown precisely until now.

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5 Concluding Remarks

In this paper we reviewed the structures and integral method of our new transformation. And exploiting our method, we calculated Feynman amplitude of vertex function. And we calculated the same integral by using Mellin-Barnes transformation method I think as one of the most beautiful calculations. In these days remarkable progress has been made concerning Feynman integral calculations. The results were represented by generalized hyper-geometric functions (Appell function, Lauricella function and Kampé de Fériet function etc.). But the concrete analytic shapes, asymptotic behaviors and recurrence relations of these functions are not clear until now. For a few years now these formulas have been derived by using the results of Feynman integral calculations as a test function. For these reasons we can't obtain meaningful numerical results of the Feynman amplitude in QCD with arbitrary accuracy,expressed in terms of hyper-geometric function. On the other hand the computer programs for the numerical calculation of hyper-geometric function have been developed. But the delicate relations among the integral domains, the kernels and the elements of the complex propagator integrals are indistinct. Furthermore we need to compare the results of the Feynman integral calculations by using several kinds of method and examine them circumstantially. At last it is important that we investigate the mathematical aspects of our new transformation precisely, too.

Appendix A

$$A = \Big[\frac{\partial(t_1, t_2, \cdots, t_n)}{\partial(x_0, x_1, \cdots, x_{n-1})} \Big]^{-1} = \begin{vmatrix} \frac{\partial x_0}{\partial t_1} & \frac{\partial x_0}{\partial t_2} & \cdots & \frac{\partial x_0}{\partial t_n} \\ \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} & \cdots & \frac{\partial x_1}{\partial t_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_{n-1}}{\partial t_1} & \frac{\partial x_{n-1}}{\partial t_2} & \cdots & \frac{\partial x_{n-1}}{\partial t_n} \end{vmatrix}$$

$$
= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{x_0} - \frac{t_1}{x_0^2} & -\frac{t_1}{x_0^2} & \cdots & -\frac{t_1}{x_0^2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{t_{n-1}}{x_0^2} & -\frac{t_{n-1}}{x_0^2} & \cdots & -\frac{t_{n-1}}{x_0^2} \end{vmatrix} = \begin{vmatrix} 0 & 0 & \cdots & 1 \\ \frac{1}{x_0} & 0 & \cdots & -\frac{t_1}{x_0^2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\frac{t_{n-1}}{x_0^2} \end{vmatrix} \tag{36}
$$

$$
= (-1)^{n+1} \begin{vmatrix} \frac{1}{x_0} & 0 & \cdots & 0 \\ 0 & \frac{1}{x_0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{x_0} \end{vmatrix} = (-1)^{n+1} \frac{1}{x_0^{n-1}}.
$$

Appendix B

We consider the following symmetric variable transformation and prove that the Jacobian becomes ∞.

$$
\Big(x_i\Big) = \Big(t_i / \sum_{i=1}^{n} t_i\Big) = \Big(t_i / x_0\Big) \tag{37}
$$

where i runs from 1 to n and $x_0 = \sum_{i=1}^{n} t_i$. We put the Jacobian of the integral transformation, from t_i to x_i, J.

$$
J^{-1} = \Big[\frac{\partial(t_1, t_2, \cdots, t_n)}{\partial(x_1, x_2, \cdots, x_n)}\Big]^{-1} = \begin{vmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} & \cdots & \frac{\partial x_1}{\partial t_n} \\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} & \cdots & \frac{\partial x_2}{\partial t_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial t_1} & \frac{\partial x_n}{\partial t_2} & \cdots & \frac{\partial x_n}{\partial t_n} \end{vmatrix} \tag{38}
$$

$$
= \begin{vmatrix} \frac{1}{x_0} - \frac{t_1}{x_0^2} & -\frac{t_1}{x_0^2} & \cdots & -\frac{t_1}{x_0^2} \\ -\frac{t_2}{x_0^2} & \frac{1}{x_0} - \frac{t_2}{x_0^2} & \cdots & -\frac{t_2}{x_0^2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{t_n}{x_0^2} & -\frac{t_n}{x_0^2} & \cdots & \frac{1}{x_0} - \frac{t_n}{x_0^2} \end{vmatrix} = \begin{vmatrix} \frac{1}{x_0} & 0 & 0 & \cdots & 0 & -\frac{t_1}{x_0^2} \\ 0 & \frac{1}{x_0} & 0 & \cdots & 0 & -\frac{t_2}{x_0^2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{x_0} & -\frac{t_{n-1}}{x_0^2} \\ -\frac{1}{x_0} & -\frac{1}{x_0} & -\frac{1}{x_0} & \cdots & -\frac{1}{x_0} & \frac{1}{x_0} - \frac{t_n}{x_0^2} \end{vmatrix}
$$

Performing the Laplace expansion of this determinant along the first row, we can obtain the following $(n-1) \times (n-1)$ determinants

$$
\frac{1}{x_0} \begin{vmatrix} \frac{1}{x_0} & 0 & \cdots & -\frac{t_2}{x_0^2} \\ 0 & \frac{1}{x_0} & \cdots & -\frac{t_3}{x_0^2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{x_0} & -\frac{1}{x_0} & \cdots & \frac{1}{x_0} - \frac{t_n}{x_0^2} \end{vmatrix} + \Big(-1\Big)^{n+1} \Big(-\frac{t_1}{x_0^2}\Big) \begin{vmatrix} 0 & \frac{1}{x_0} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{x_0} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{x_0} \\ -\frac{1}{x_0} & -\frac{1}{x_0} & -\frac{1}{x_0} & \cdots & -\frac{1}{x_0} & -\frac{1}{x_0} \end{vmatrix}
$$

$$
= \Big(A-1\Big) + \Big(B-1\Big). \tag{39}
$$

Shifting the first column of $\Big(B-1\Big)$ to the last column, we can make a triangular determinant and calculate the determinant easily. That is,

$$
\Big(B-1\Big) = \Big(-1\Big)^{n+1} \Big(-\frac{t_1}{x_0^2}\Big) \Big(-1\Big)^{n-2} \begin{vmatrix} \frac{1}{x_0} & 0 & \cdots & 0 \\ 0 & \frac{1}{x_0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{x_0} & -\frac{1}{x_0} & \cdots & -\frac{1}{x_0} \end{vmatrix}
$$

$$= -\frac{t_1}{x_0^{n+1}}. \tag{40}$$

Applying the same technique to $\Big(A-1\Big)$,we can obtain the $(n-2)\times(n-2)$ determinants.

$$\Big(A-1\Big) = \frac{1}{x_0^2}\begin{vmatrix} \frac{1}{x_0} & 0 & \cdots & -\frac{t_3}{x_0^2} \\ 0 & \frac{1}{x_0} & \cdots & -\frac{t_4}{x_0^2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{x_0} & -\frac{1}{x_0} & \cdots & \frac{1}{x_0}-\frac{t_n}{x_0^2} \end{vmatrix}$$

$$+\frac{1}{x_0}\Big(-1\Big)^n\Big(-\frac{t_2}{x_0^2}\Big)\begin{vmatrix} 0 & \frac{1}{x_0} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{x_0} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{x_0} \\ -\frac{1}{x_0} & -\frac{1}{x_0} & -\frac{1}{x_0} & \cdots & -\frac{1}{x_0} & -\frac{1}{x_0} \end{vmatrix}$$

$$= \Big(A-2\Big) + \Big(B-2\Big) \tag{41}$$

Following the same procedure to $\Big(B-2\Big)$ as $(B-1\Big)$, we have the result as

$$\Big(B-2\Big) = -\frac{t_2}{x_0^{n+1}} \tag{42}$$

We calculate the determinants in the same way successively. Then we can get the final result as follows:

$$J^{-1} = \frac{1}{x_0^{n-2}}\begin{vmatrix} \frac{1}{x_0} & -\frac{t_{n-1}}{x_0^2} \\ -\frac{1}{x_0} & \frac{1}{x_0}-\frac{t_n}{x_0^2} \end{vmatrix} - \sum_{i=1}^{n-2}\Big(B-i\Big) \tag{43}$$

$$= \frac{1}{x_0^n} - \frac{1}{x_0^{n+1}}\sum_{i=1}^{n} t_i = \frac{1}{x_0^n} - \frac{1}{x_0^n} = 0$$

Therefore the Jacobian J becomes ∞.

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