Supplements to the assertion in the previous paper

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1 Introduction

I contributed a paper in Hokkaido Math.J. (1981) vol.10 . Recently, I've been felt some criticisms for it's paper. These are entirely groundless. Hence this paper has become to describe. Therefore, this paper is the supplements of previous paper. It is the first time that the contents of this paper is published. However I had already know facts in this paper since nearly more twenty years ago more. Since there are some complicated circumstances at that time, only I couldn't do so.

In first stage , a primitve interesting object for me is a cubic basis. Let $\{C_1, C_2, C_3\}$ be the subset of degree 3 symmmetric matrices. As we say that $\{C_1, C_2, C_3\}$ is cubic basis, these degree 3 symmmetric matrices are linearly independent and satisfied the conditions $C_ie_j = C_je_i,\ (i, j = 1, 2, 3)$, where $e_i\ (i = 1, 2, 3)$ is a canonical basis in the column vector space $\mathbf{R}^3$. Given any 3-dimentional subspace α in the vector space $S(3)$ that consists of all degree 3 symmetric matrices, I ask whether a cubic basis exists on α. For concretely given α , we can be easily dicide the existance of cubic basis. However given any subspace α have some parameters, the answer is not trivial.

For example

$$\alpha = \langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & b & d \\ 0 & d & c \end{pmatrix} \rangle.$$

In this subspace does a cubic basis exist? Moreover does a cubic basis exist under what necessary and sufficient condition?

Beyond three degree, similarly we can consider the problem of existance of the cubic basis. I think we should set the problem to in what dimensional subspace a cubic basis certainly exist, so that it may become mathimatical meaningful problem.

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In this paper, we slightly deform some symmbols in the previous paper, but these are not essential change. It's deformation is natural for one who already had known the previous paper. Finally, to my wife Takiko, I'll want express my best thanks for supports during thirty years more.

2 Assertion

Let $S(3)$ be the set of all 3-symmetric matrices. Let e_i ($i = 1, 2, 3$) be the canonical column vectors in $\mathbf{R}^3$. We will set the 3-subspace α_t in vector space $S(3)$ to be generated by symmetric matrices $Q(e_1, e_2)$, $Q(e_1, e_3) - 2P(e_2)$ and $P(e_1)$ - $tP(e_3)$ where t is a positive parameter. When these notations are used for the canonical base in $\mathbf{R}^3$ we will use to symplify the notation Q_{12} , $Q_{13} - 2P_2$, $P_1 - tP_3$. Let $\alpha_t^\perp$ to be the orthogonal complement of 3-subspace α_t in $S(3)$. Let β be the 2-subspace of α_t which consists of two symmetric matrices Q_{12} and $Q_{13} - 2P_2$. When U is a element of $SL(3)$, we set $U = (u_1, u_2, u_3)$ where u_i ($i = 1.2.3$) are column vectors in $\mathbf{R}^3$. Let F_U be the corresponce of a symmetric matrix A to a symmetric matrix UA^tU. F_U induces the diffeomorphism of the grassmannian manifold $G_k(S(3))$ which consists of all k-subspaces of $S(3)$. In this case , we'll donote these to same F_U . Generally ,for $\delta \in G_k(S(3))$, we denote the isotropy group $\{U \in SL(3) \mid F_U\delta = \delta\}$ to $Iso(\delta)$. Let $Iso_0(\delta)$ be it's connected compoment includes the identity matrix. The $\mathfrak{J}(\delta)$ is the Lie algebra of $Iso_0(\delta)$. Simmilary, we denote the closure in $G_k(S(3))$ of orbit $\{ F_U\delta \in G_k(S(3)) \mid U \in SL(3) \}$ of δ to $cl(orb(\delta))$. On the above β, we remark that any element $\gamma \in orb(\beta)$ is represented by the 2-subspace which is generated by two symmtric matrices $Q(u_1, u_2), Q(u_1, u_3) - 2P(u_2)$, for some $(u_1, u_2.u_3) \in SL(3)$.

Assertion For every positive number t, $G_2(\alpha_t) \cap cl(orb(\beta)) = G_2(\alpha_t) \cap orb(\beta)$, and this subset have three elements. At each intersections, the $orb(\beta)$ cross transversely to the $G_2(\alpha_t)$.

Proof The subspace α_t excludes any 1-projection matrix $P(v)$ (where v is any unit colomn 3-vector) and the tangent space of $\{$ line $\langle P(v)\rangle \mid$ any v is colomn 3-vector $\}$ at $\langle P(v)\rangle$ ($\subset$ the tangent space of $G_1(S(3))$ at $\langle P(v)\rangle$) . The former case is clear from the poperty of 1-projection matrix $P(v)$. The latter case is shown that the subspace $\alpha_t^\perp$ also excludes any 1-projection matrix $P(v)$. From the structure theorem of $cl(orb(\beta))$, we recognize $G_2(\alpha_t) \cap cl(orb(\beta)) = G_2(\alpha_t) \cap orb(\beta)$.

From the condition $\gamma \subsetneq \alpha_t$, we get the following eqations for any $A \in \alpha_t^\perp$:

$$trAQ(u_1, u_2) = 0 \tag{1}$$

$$trA(Q(u_1, u_3) - 2P(u_2)) = 0 \ . \tag{2}$$

From the trace formmla , deform above expressions the following

$${}^tu_1Au_2 = 0 \tag{3}$$

$$ {}^{t}u_1 A u_3 = {}^{t}u_2 A u_2 \quad . \tag{4} $$

Since there is a cubic basis on $\alpha_t^{\perp}$, we put these basis to C_i $(i = 1, 2, 3)$. Using this basis , rewrite the upper equations.

We obtain the following equations:

$$ A_1 u_2 = 0 \, , or \; A_2 u_1 = 0 \, , \tag{5} $$

$$ A_1 u_3 = A_2 u_2 \, , \tag{6} $$

where $A_i = \sum_{j=1}^{3} u_{ji} C_j$, and $u_i = {}^{t}(u_{1i}, u_{2i}, u_{3i})$.
From (5),(6), we can determine the vector u_2 up to scaler multiple. For the vector u_2, the next expressions must be satisfied.

$$ det A_2 = 0 \, , \tag{7} $$

$$ {}^{t}u_2 A_2 u_2 = 0 \, . \tag{8} $$

Also from (5), vector u_1 is determined except scalar multiple. Similary up to scalar multiple, u_3 is obtained with (6). Explicit expressions of u_i $(i = 1, 2, 3)$ are next ,where signs are respectively subject to order.

$$ u_1 = \begin{pmatrix} -1/(\lambda\sqrt{t}) \\ \mp\mu/\sqrt[4]{t} \\ 1 \end{pmatrix} u_{31} \, , \quad or \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u_{11}. \tag{9} $$

$$ u_2 = \begin{pmatrix} \lambda/\sqrt{t} \\ \pm\mu/\sqrt[4]{t} \\ 1 \end{pmatrix} u_{32} \, , \quad or \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_{22}. \tag{10} $$

$$ u_3 = \begin{pmatrix} \lambda/\sqrt{t} \\ \pm\mu/\sqrt[4]{t} \\ 1 \end{pmatrix} u_{33} + \begin{pmatrix} -\lambda(\lambda^2+1)/\sqrt{t} \\ \pm 2\mu/\sqrt[4]{t} \\ 0 \end{pmatrix} u_{32}^2/u_{31} \, , \quad or \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_{23} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_{22}^2/u_{11} \tag{11} $$

where $\lambda = -\sqrt{2\sqrt{3}-3}$, $\mu = \sqrt[4]{4(2/\sqrt{3}-1)}$.
This is a solution of the following simaltaneous equations :

$$ \lambda^2 - \lambda\mu^2 - 1 = 0 \, , \; \lambda^3 + 3\mu^2 + 3\lambda = 0 \, . \tag{12} $$

Easily, this simaltaneous equations are solvable. That is

$$ \lambda^4 + 6\lambda^2 - 3 = 0 \, , \; \mu^2 = -2\lambda/\sqrt{3} \, . $$

Then we have had the three element of $orb(\beta)$ in α_t for each positive t. The followings are the explicit representation of three 2-subspaces:

$$\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \rangle$$

$$\langle \begin{pmatrix} -2/\sqrt{t} & \mp\mu(\lambda^2+1)/(\lambda\sqrt[4]{t}) & \mu^2 \\ \mp\mu(\lambda^2+1)/(\lambda\sqrt[4]{t}) & -2\mu^2 & 0 \\ \mu^2 & 0 & 2\sqrt{t} \end{pmatrix} ,$$

$$\begin{pmatrix} 2/\sqrt{t} & \pm\mu(\lambda^4-\lambda^2-2)/(\lambda\sqrt[4]{t}) & 3\mu^2 \\ \pm\mu(\lambda^4-\lambda^2-2)/(\lambda\sqrt[4]{t}) & -6\mu^2 & 0 \\ 3\mu^2 & 0 & -2\sqrt{t} \end{pmatrix} \rangle$$

Scalar multiples u_{31}, u_{32}, u_{33} (u_{11}, u_{22}, u_{23} in trivial case) do not influence to determine the each 3-subspaces. More pricisely, these scalars arise from the isotropy group $Iso(\beta)$. Indeed the dimension of $Iso(\beta)$ is three in $GL(3)$, or two in $SL(3)$. The next property is the condition of $U \in SL(3)$. In this paper , this fact essentially is not used, but I'll mention. Since $\det U = 1$, we get

$$u_{32} = \pm\nu\sqrt[4]{t} \ , \ where \ \ \nu^3 = \lambda/(2(\lambda^2+1)^2\mu) \ (<0) \ .$$

We'll show the transversality in each intersections. It may be assumed that the 3-subspace $F_{U^{-1}}(\alpha_t)$ is generated by 2-subspace β and the following symmetric matrix B:

$$B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & b_4 \\ 0 & b_4 & b_3 \end{pmatrix}$$

If the intersection is trivial then U = identity matrix I and $b_1 = 1$, $b_2 = b_4 = 0$ and b_3 = -t. And so we'll considered the case $U \neq I$.

Lemma If we set $u_{32}^2/u_{31} = (-(\lambda^2+7)/2)u_{33}$, then $b_1 = 1$, $b_3 = s^2/k_1$. $b_2 = b_4 = 0$, where $s = u_{31}/u_{33}$, $k_1 = -(\lambda^2+6)$.

Proof of lemma Let ${}^tU = (v_1, v_2, v_3)$. First,we'll determine the subspace $F_{{}^tU}(\alpha_t^\perp)$, this is achieved by caluculating the elements $tP(v_1) + P(v_3)$, $Q(v_2, v_3)$, $Q(v_1, v_3) + P(v_2)$. Using the exprssion (12), we have followings:

$$tP(v_1) + P(v_3) = (\lambda^2+1)\begin{pmatrix} u_{31}^2/\lambda^2 & 0 & u_{32}^2 \\ 0 & u_{32}^2 & u_{32}(u_{33} - \lambda^2 u_{32}^2/u_{31}) \\ u_{32}^2 & u_{32}(u_{33} - \lambda^2 u_{32}^2/u_{31}) & (u_{33} - \lambda^2 u_{32}^2/u_{31})^2 + \lambda^2 u_{32}^4/u_{31}^2 \end{pmatrix}$$

$$Q(v_2, v_3) = \pm 2\mu/\sqrt[4]{t}\begin{pmatrix} -u_{31}^2 & 0 & u_{32}^2 \\ 0 & u_{32}^2 & u_{32}(u_{33} + u_{32}^2/u_{31}) \\ u_{32}^2 & u_{32}(u_{33} + u_{32}^2/u_{31}) & u_{33}^2 + 2u_{32}^2 u_{33}/u_{31} \end{pmatrix}$$

$$Q(v_1,v_3)+P(v_2)=1/\sqrt{t}\begin{pmatrix} (\mu^2-2/\lambda)u_{31}^2 & 0 & (\mu^2+2\lambda)u_{32}^2 \\ 0 & (\mu^2+2\lambda)u_{32}^2 & u_{32}\{\,(\mu^2+2\lambda)u_{33}+(2\mu^2-\lambda^3-\lambda)u_{32}^2/u_{31}\,\} \\ (\mu^2+2\lambda)u_{32}^2 & u_{32}\{\,(\mu^2+2\lambda)u_{33}+(2\mu^2-\lambda^3-\lambda)u_{32}^2/u_{31}\,\} & (\mu^2+2\lambda)u_{33}^2+2(2\mu^2-\lambda^3-\lambda)u_{32}^2u_{33}/u_{31}+4\mu^2u_{32}^4/u_{31}^2 \end{pmatrix}$$

We exchange the basis and substitute $u_{32}^2/u_{31}=(-(\lambda^2+7)/2)u_{33}$. We get followings:

$$\begin{pmatrix} -u_{31}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_1u_{33}^2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

From $F_{U^{-1}}(\alpha_t)=(F_{{}^tU}(\alpha_t^{\perp}))^{\perp}$, we obtain the matix B of lemma.
(Q.E.D of the lemma)

Let be $\delta_1=\langle B,P_2,P_3,Q_{23}\rangle$, and $\delta_0=\langle P_2,P_3,Q_{23}\rangle$. As for the local coordinate of manifold $G_2(S(3))$ at the β ,we may take the $hom(\beta,\delta_1)$. Furthermore we can identify it with tangent space of $G_2(S(3))$ at β. Then in order to show the transversality at β of $orb(\beta)$ with $G_2(\langle\beta,B\rangle)$, we may only understand that the tangent space at β of $orb(\beta)$ is span $hom(\beta,\delta_0)$ modulo to the subspace $F_{U^{-1}}(\alpha_t)$. We'll recall the clasification theorem of $cl(orb(\beta))$, the Lie algebra $\mathfrak{I}(\beta)$ of the isotropy group of β is following:

$$\mathfrak{I}(\beta)=\left\{\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & -a_{11} \end{pmatrix}\right\}$$

Then we may only examine f_a for under the matrix a.

$$a=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & -2a_{11} & 0 \\ a_{31} & a_{32} & a_{11} \end{pmatrix}$$

f_a $(x_1Q_{12}+x_2(Q_{13}-2P_2)$ modulo to $F_{U^{-1}}(\alpha_t)$ is;

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2(a_{21}+a_{32}-a_{12}b_2) & a_{31} \\ 0 & a_{31} & -2a_{12}b_3 \end{pmatrix}x_1+\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2(6a_{11}-a_{13}b_2) & a_{21}-2a_{32} \\ 0 & a_{21}-2a_{32} & 2(a_{31}-a_{13}b_3) \end{pmatrix}x_2$$

Then if $b_3\neq 0$, this span the $hom(\beta,\delta_0)$. This is completed the assertion.
(Q.E.D)

3 A conclusion induced by the assertion

We put the orbit of $\beta=\langle Q_{12}\ ,Q_{13}-2P_2\rangle$ by the group of nonsigular upper traiangular matrices to $\{\ \beta(a_1,a_2,t)\mid a_1,a_2,t>0\ \}$, where

$$\beta(a_1,a_2,t)\ =\ \langle a_1P_1+Q_{12},\ a_2P_1+t(P_1-P_2)+Q_{13}\rangle.$$

When $a_1 = a_2 = 0$, for simplicfication of the notation we denote $\beta(0,0,t)$ to β_t. Easily the followings will be understood (besides we remark that the same properties is ture for non zero t):

$$orb(\beta) = \bigcup \{ \ SO(3)\ orbit\ of\ \beta(a_1,a_2,t) \ \mid\ any\ a_1, any\ a_2,\ t > 0 \ \},$$

and

$$orb(\beta) \bigcap G_2(\ \langle I \rangle^{\perp}) = \bigcup_{t>0} \{ \ SO(3) \text{ orbit of } \beta_t \ \}.$$

Proposition 3.1 If a symmetric matrix A in β_t has the multiple eigen values, then $A = (1/\sqrt{2})(P_1 - P_2) + Q_{13}$ up to scalar multiple.

Proof The characteristic equation of the problem is following:

$$\begin{vmatrix} t-\lambda & x & 1 \\ x & -t-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

where any $x, t > 0$. The condition of multiple eigenvalues is $(x^2 + t^2 + 1)/3 - \sqrt[3]{t^2/4} \ = \ 0$. This expression is deform to $x^2/3 + (1/3)(\sqrt[3]{t^2/4} + 1)(2\sqrt[3]{t^2/4} - 1)^2 \ = \ 0$. Then we get the conclusion: i.e. $x = 0,\ \ t^2 = 1/2$.
(Q.E.D)

Proposition 3.2 If a symmetric matrix A in $\beta(a_1, a_2, t)$ has trace zero and the multiple eigen values, then $a_2 = 0$ and $A = (1/\sqrt{2})(P_1 - P_2) + Q_{13}$ up to scalar multiple.

Proof When $a_1 \neq 0$, from the trace condition of A we get $A = -(a_2/a_1)Q_{12} + t(P_1 - P_2) + Q_{13}$. From the proposition3.1 we take $a_2 = 0$ and $t = 1/\sqrt{2}$. The otherwise when $a_1 = 0$, from the trace condition of A we get $a_2 = 0$ too.
(Q.E.D)

Let $\Sigma \ = \{ \ \alpha \mid P_1 + P_2 - 2P_3 \in \alpha \ , \ \alpha \in orb(\beta) \}$. Then there is some $U \in SO(3)$ and some $A \in \beta(a_1, a_2.t)$, we have $P_1 + P_2 - 2P_3 = F_U A$. Applying upper Propositions to A, we have $a_2 = 0$ and $t = 1/\sqrt{2}$. All elements $(u_1, u_2, e_3) \in SO(3)$ preserve the matrix $P_1 + P_2 - 2P_3$ are of a subgroup of $SO(3)$, this subgroup is isomorphic to $SO(2)$. Let $\bar{\Sigma} \ = \ \{ \ \alpha \mid P_1 + P_2 - 2P_3 \in \alpha, \alpha \in cl(orb(\beta)) \ \}$. If $\alpha \in cl(orb(\beta)) - orb(\beta)$, then from the structure theorem of $cl(orb(\beta))$, we obtain

$$\alpha = \langle \ P(u_1),\ a_1 Q(u_1,u_2) + t(P(u_1) - P(u_2)) + Q(u_1,u_3) \ \ (t > 0) \ \rangle$$

for $(u_1, u_2, u_3) \in SO(3)$. Applying the proposition 3.1 to $\alpha \in \bar{\Sigma}$, we obtain $a_1 = 0$ and $t = 1/\sqrt{2}$. If $\gamma \in \bar{\Sigma} - \Sigma$ then $\gamma = F_U \alpha$ for $\alpha = \langle P(u_1)\ ,\ P_1 + P_2 - 2P_3 \rangle$, where $U = (u_1, u_2, u_3) \in SO(3)$ and U satisfies the following condition :

$$P_1 + P_2 - 2P_3 = -\sqrt{2}(\ (1/\sqrt{2})(P(u_1) - P(u_2) + Q(u_1, u_3)).$$

Therefore any element α of $\bar{\Sigma}$ represent as $\langle\ P(u_1)\sin\phi + Q(u_1, u_2)\cos\phi\ ,\ P_1 + P_2 - 2P_3 \rangle$ where $|\phi| \leq \pi/2$.

Proposition 3.3 $\bar{\Sigma}$ is deffeomorphic to 2-torus.

Proof Let

$$V = \begin{pmatrix} 1/\sqrt{3} & 0 & -\sqrt{2/3} \\ 0 & 1 & 0 \\ \sqrt{2/3} & 0 & 1/\sqrt{3} \end{pmatrix}.$$

Then the orthogonal matix V satisfies the condition : $F_V(\ (1/\sqrt{2})(P_1 - P_2) + Q_{13}) = -(1/\sqrt{2})(P_1 + P_2 - 2P_3)$. Let $A = F_V(P_1 \sin\phi + Q_{12}\cos\phi)$ for $|\phi| \leq \pi/2$. Let $W = (w_1, w_2, e_3)$ be a orthogonal matrix. It is saisfied that $F_W(P_1 + P_2 - 2P_3) = P_1 + P_2 - 2P_3$ by each $W = (w_1, w_2, e_3)$. The set consist of this W is homomophic to $SO(2)$ and is parametrized by $\theta(mod.2\pi)$. Let any $\gamma \in \bar{\Sigma}$ be represented by $\langle F_W A\ , P_1 + P_2 - 2P_3\rangle$. We can check $(\phi(mod.\pi), \theta(mod.\pi))$ is a chart of $\bar{\Sigma}$.
(Q.E.D)

Definition A linear transformation between symmetric matices $\rho : S(3) \longrightarrow \langle P_3\rangle^{\perp}$ is defined by $A \longmapsto A + (1/2)(trAP_3)(P_1 + P_2 - 2P_3)$. The linear transformation ρ induces the map from $cl(orb(\beta)) - \bar{\Sigma}$ to $G_2(\ \langle P_3\rangle^{\perp})$. For this induced map, we use the same symbol ρ whenever without the confusion.

Proposition 3.4 The map ρ from $cl(orb(\beta)) - \bar{\Sigma}$ to $G_2(\ \langle P_3\rangle^{\perp})$ is extendable to the map $\tilde{\rho}$ on $\sigma_{\bar{\Sigma}}(cl(orb(\beta)))$.

Proof Let $\nu(\Sigma)$ be the normal bundle of 2-submanifold Σ in 6-manifold $orb(\beta)$. From the structure theorem, One dimentional manifold $\bar{\Sigma} - \Sigma\ (\subset \Sigma)$ is contained within a stratum of the stratified set $cl(orb(\beta))$. Let $\nu(\bar{\Sigma} - \Sigma)$ be the normal bundle of $\bar{\Sigma} - \Sigma$ in this stratum which is a 5-manifold. Then let ι be the projection of total space $E(\nu(\bar{\Sigma}))$ to base space $\bar{\Sigma}$. We consider the blowing up $cl(orb(\beta))$ at each points of $\bar{\Sigma}$ that we denote to $\sigma_{\bar{\Sigma}}(cl(orb(\beta)))$. To make this process we must take a coordinate neighbourhood of each $\alpha \in \Sigma$. For any $\alpha \in \bar{\Sigma}$, we can put $\alpha = \langle\ P(u_1)\sin\phi + Q(u_1, u_2)\cos\phi\ , P_1 + P_2 - 2P_3\rangle$, where

$$P_1 + P_2 - 2P_3 = -\sqrt{2}(\ (1/\sqrt{2})(P(u_1) - P(u_2)) + Q(u_1, u_3)\ )$$

and $U = (u_1, u_2, u_3) \in SO(3)$.

The fiber $\iota^{-1}(\alpha)$ on α is generated by the derivative of the following 2-subspace $\alpha_t \in orb(\beta)$. Let $\alpha_t = \langle A_t\ , B_t\rangle$, where A_t, B_t are as follows :

$$\begin{aligned} A_t &= P(X_t u_1)\sin\phi + Q(X_t u_1, X_t u_2)\cos\phi, \\ B_t &= tx_1(\ P(X_t u_1)\cos\phi - Q(X_t u_1, X_t u_2)\sin\phi) + tx_2 P(X_t u_2) + \\ &\qquad (1/\sqrt{2})(P(X_t u_1) - P(X_t u_2)) + Q(X_t u_1, X_t u_3)\ , \end{aligned}$$

where $X_t = exp(tX)$, $X \notin \mathfrak{Z}(\alpha_0)$ and

$$X = \begin{pmatrix} 0 & 0 & x_3 \\ 0 & 0 & x_4 \\ -x_3 & -x_4 & 0 \end{pmatrix}.$$

We remark $\alpha = \alpha_0$, and $\rho(B_0) = 0$. Besides, $\rho(\alpha_t) = \langle \rho(A_t)\ ,\ \rho(B_t) \rangle = \langle \rho(A_t)\ , \rho((B_t - B_0)/t)\ \rangle$. We define $\tilde{\rho}(\alpha_0) = lim_{t\to 0}\rho(\alpha_t)$.
Therefore $\tilde{\rho}(\alpha_0) = \langle\ \rho(P(u_1)\sin\phi + Q(u_1, u_2)\cos\phi)\ , \rho(d/dt\,|_{t=0}B_t)\ \rangle$. The derivative $d/dt\,|_{t=0}B_t$ is homogeneous with the variables x_i ($i = 1, 2, 3, 4$). This map can be considered from $\sigma_\alpha(cl(orb(\beta)))$ which blows up $cl(orb(\beta))$ at α along the fiber $\iota^{-1}(\alpha)$, to $G_2(\ \langle P_3\rangle^\perp)$. Continuing this process at each point of $\bar{\Sigma}$, we obtain the extended continuous map $\tilde{\rho}$ of $\sigma_{\bar{\Sigma}}(cl(orb(\beta)))$ to $G_2(\ \langle P_3\rangle^\perp)$. The subset $\sigma_\Sigma(orb(\beta))$ is a open dense differential manifold in $\sigma_{\bar{\Sigma}}(cl(orb(\beta)))$. The restriction of $\tilde{\rho}$ on the differential manifold $\sigma_\Sigma(orb(\beta))$ is a differetial extention of ρ.
(Q.E.D)

Proposition 3.5 $\sigma_\Sigma(orb(\beta))$ is a open dense manifold in $\sigma_{\bar{\Sigma}}(cl(orb(\beta)))$. The restriction of map $\tilde{\rho}$ on $\sigma_\Sigma(orb(\beta))$ is a degree one to modulo 2 .

Proof We may identify $\sigma_{\bar{\Sigma}}(cl(orb(\beta)) = (cl(orb(\beta)) - \bar{\Sigma}) \cup (\mathbf{R}P(\nu_0(\bar{\Sigma})))$, where $\nu_0(\bar{\Sigma}) = \nu(\bar{\Sigma}) - \bar{\Sigma}$, $\mathbf{R}P(E_0)$ is the projective space bundle of vector bundle E without 0-section. For any $\alpha \in G_2(\ \langle P_3\rangle^\perp) - \tilde{\rho}(\mathbf{R}P(\nu_0(\bar{\Sigma}))$, we know $\tilde{\rho}^{-1}(\alpha) \in G_2(\langle P_1 + P_2 - 2P_3,\ \alpha\rangle) \cap cl(orb(\beta))$. Appiying the assertion to $G_2(\langle P_1 + P_2 - 2P_3,\ \alpha\rangle) \cap cl(orb(\beta))$, we obtain that $\tilde{\rho}$ is surjective. $\sigma_\Sigma(orb(\beta))$ is a open dense manifold in $\sigma_{\bar{\Sigma}}(cl(orb(\beta)))$. The restriction of $\tilde{\rho}$ on the open manifold $\sigma_\Sigma(orb(\beta))$ is differential. We use the Sard's theorem to recognize the following fact. The $\tilde{\rho}^{-1}(\alpha)$ is transversally intersect $orb(\beta)$ for almost all α. The assertion say that this intersetion number is one to modulus 2 too. Therefore we get the conclusion.
(Q.E.D)

Proposition 3.6 The map $\tilde{\rho}$ is homotopic to collapsing map τ of $\sigma_{\bar{\Sigma}}(cl(orb(\beta)))$ to $cl(orb(\beta))$.

Proof The proof is simple. We may consider $(1-t)\tilde{\rho} + t\tau$ for the homotpy. If It is conclued that $((1-t)\tilde{\rho} + t\tau)A \neq 0$ for any symmetric matrix $A(\neq 0) \in \alpha\ \ (\in cl(orb(\beta)) - \bar{\Sigma})$, then this map is well defined. $((1-t)\tilde{\rho} + t\tau)A = A + ((1-t)/2)tr(P_3A)(P_1 + P_2 - 2P_3)$ gives $((1-t)\tilde{\rho} + t\tau)A \neq 0$.
(Q.E.D)

We easily obtain the following theorem, from a series of upper propositions.

Theorem The homology class $[cl(orb(\beta))]$ is homologous to the fundamental class $[G_2(\ \langle P_3\rangle^\perp)]$ in $H_6(\ G_2(S(3)) : \mathbf{Z}_2)$.