Supplements to the assertion in the previous paper

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1 Introduction

I contributed a paper in Hokkaido Math.J. (1981) vol.10. Recently, I’ve been felt some criticisms for it’s paper. These are entirely groundless. Hence this paper has become to describe. Therefore, this paper is the supplements of previous paper. It is the first time that the contents of this paper is published. However I had already know facts in this paper since nearly more twenty years ago more. Since there are some complicated circumstances at that time, only I couldn’t do so.

In first stage, a primitive interesting object for me is a cubic basis. Let \{C_1, C_2, C_3\} be the subset of degree 3 symmetric matrices. As we say that \{C_1, C_2, C_3\} is cubic basis, these degree 3 symmetric matrices are linearly independent and satisfied the conditions \(C_i e_j = C_j e_i\) (\(i, j = 1, 2, 3\)), where \(e_i\) (\(i = 1, 2, 3\)) is a canonical basis in the column vector space \(\mathbb{R}^3\). Given any 3-dimentional subspace \(\alpha\) in the vector space \(S(3)\) that consists of all degree 3 symmetric matrices, I ask whether a cubic basis exists on \(\alpha\). For concretely given \(\alpha\), we can be easily decide the existance of cubic basis. However given any subspace \(\alpha\) have some parameters, the answer is not trivial.

For example

\[
\alpha = \langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & b & d \\ 0 & d & c \end{pmatrix} \rangle.
\]

In this subspace does a cubic basis exist? Moreover does a cubic basis exist under what necessary and sufficient condition?

Beyond three degree, similarly we can consider the problem of existance of the cubic basis. I think we should set the problem to in what dimensional subspace a cubic basis certainly exist, so that it may become mathimatical meaningful problem.
In this paper, we slightly deform some symbols in the previous paper, but these are not essential change. It’s deformation is natural for one who already had known the previous paper. Finally, to my wife Takiko, I’ll want express my best thanks for supports during thirty years more.

2 Assertion

Let $S(3)$ be the set of all 3-symmetric matrices. Let $e_i$ $(i = 1, 2, 3)$ be the canonical column vectors in $\mathbb{R}^3$. We will set the 3-subspace $\alpha_i$ in vector space $S(3)$ to be generated by symmetric matrices $Q(e_1, e_2)$, $Q(e_1, e_3) - 2P(e_2)$ and $P(e_1) - tP(e_3)$ where $t$ is a positive parameter. When these notations are used for the canonical base in $\mathbb{R}^3$ we will use to simplify the notation $Q_{12}$, $Q_{13} - 2P_2$, $P_1 - tP_3$. Let $\alpha_i^\perp$ to be the orthogonal complement of 3-subspace $\alpha_i$ in $S(3)$. Let $\beta$ be the 2-subspace of $\alpha_i$ which consists of two symmetric matrices $Q_{12}$ and $Q_{13} - 2P_2$. When $U$ is a element of $SL(3)$, we set $U = (u_1, u_2, u_3)$ where $u_i$ $(i = 1, 2, 3)$ are column vectors in $\mathbb{R}^3$. Let $F_{U}$ be the correspondence of a symmetric matrix $A$ to a symmetric matrix $UAU$. $F_{U}$ induces the diffeomorphism of the grassmannian manifold $G_k(S(3))$ which consists of all $k$-subspaces of $S(3)$. In this case, we’ll denote these to same $F_{U}$. Generally, for $\delta \in G_k(S(3))$, we denote the isotropy group \{ $U \in SL(3)$ $|$ $F_{U}\delta = \delta$ \} to $Iso(\delta)$. Let $Iso_0(\delta)$ be it’s connected component includes the identity matrix. The $3(\delta)$ is the Lie algebra of $Iso_0(\delta)$. Simmillary, we denote the closure in $G_k(S(3))$ of orbit \{ $F_{U}\delta \in G_k(S(3))$ $|$ $U \in SL(3)$ \} of $\delta$ to $cl(orb(\delta))$. On the above, we remark that any element $\gamma \in orb(\beta)$ is represented by the 2-subspace which is generated by two symmtric matrices $Q(u_1, u_2)$, $Q(u_1, u_3) - 2P(u_2)$, for some $(u_1, u_2, u_3) \in SL(3)$.

Assertion For every positive number $t$, $G_2(\alpha_i) \cap cl(orb(\beta)) = G_2(\alpha_i) \cap orb(\beta)$, and this subset have three elements. At each intersections, the $orb(\beta)$ cross transversely to the $G_2(\alpha_i)$.

Proof The subspace $\alpha_i$ excludes any 1-projection matrix $P(v)$ (where $v$ is any unit column 3-vector) and the tangent space of \{ line $\langle P(v) \rangle$ $|$ any $v$ is column 3-vector \} at $\langle P(v) \rangle$ ( $\subset$ the tangent space of $G_1(S(3))$ at $\langle P(v) \rangle$ ) . The former case is clear from the propety of 1-projection matrix $P(v)$. The latter case is shown that the subspace $\alpha_i^\perp$ also excludes any 1-projection matrix $P(v)$. From the structure theorem of $cl(orb(\beta))$, we recognize $G_2(\alpha_i) \cap cl(orb(\beta)) = G_2(\alpha_i) \cap orb(\beta)$.

From the condition $\gamma \subseteq \alpha_i$, we get the following equations for any $A \in \alpha_i^\perp$:

\[
trAQ(u_1, u_2) = 0 \tag{1}
\]

\[
trA(Q(u_1, u_3) - 2P(u_2)) = 0 . \tag{2}
\]

From the trace formula, deform above expressions the following

\[
^t u_1Au_2 = 0 \tag{3}
\]
\[ t_u^1 A u_3 = t_u^2 A u_2 . \]  

(4)

Since there is a cubic basis on \( \alpha^+_t \), we put these basis to \( C_i \) \( (i = 1, 2, 3) \). Using this basis, rewrite the upper equations.

We obtain the following equations:

\[ A_1 u_2 = 0, \text{or} \ A_2 u_1 = 0 , \]  

(5)

\[ A_1 u_3 = A_2 u_2 , \]  

(6)

where \( A_i = \sum_{j=1}^3 u_j C_j \), and \( u_i = t(u_{1i}, u_{2i}, u_{3i}) \).

From (5), (6), we can determine the vector \( u_2 \) up to scalar multiple. For the vector \( u_2 \), the next expressions must be satisfied.

\[ \det A_2 = 0 , \]  

(7)

\[ t_u^1 A_2 u_2 = 0 . \]  

(8)

Also from (5), vector \( u_1 \) is determined except scalar multiple. Similarly up to scalar multiple, \( u_3 \) is obtained with (6). Explicit expressions of \( u_i \) \( (i = 1, 2, 3) \) are next, where signs are respectively subject to order.

\[ u_1 = \begin{pmatrix} -1/(\lambda \sqrt{t}) \\ \mp \mu/\sqrt{3} \\ 1 \end{pmatrix} u_{31}, \text{ or } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u_{11} . \]  

(9)

\[ u_2 = \begin{pmatrix} \lambda/\sqrt{3} \\ \pm \mu/\sqrt{3} \\ 1 \end{pmatrix} u_{32}, \text{ or } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_{22} . \]  

(10)

\[ u_3 = \begin{pmatrix} \lambda/\sqrt{3} \\ \pm \mu/\sqrt{3} \\ 1 \end{pmatrix} u_{33} + \begin{pmatrix} -\lambda(\lambda^2 + 1)/\sqrt{3} \\ \pm 2\mu/\sqrt{3} \\ 0 \end{pmatrix} u_{32}^2/u_{31} , \text{ or } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_{23} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_{22}^2/u_{11} . \]  

(11)

where \( \lambda = -\sqrt{2 \sqrt{3} - 3} , \mu = \sqrt{4(2/\sqrt{3} - 1)} \).

This is a solution of the following simultaneous equations:

\[ \lambda^2 - \lambda^2 - 1 = 0 , \lambda^3 + 3\mu^2 + 3\lambda = 0 . \]  

(12)

Easily, this simultaneous equations are solvable. That is

\[ \lambda^4 + 6\lambda^2 - 3 = 0 , \mu^2 = -2\lambda/\sqrt{3} . \]
Then we have had the three element of orb(β) in αt for each positive t. The followings are the explicit representation of three 2-subspaces:

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 & 1 \\
0 & -2 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

\[
\langle \begin{pmatrix}
-2/\sqrt{t} \\
\mp\mu(\lambda^2 + 1)/(\lambda \sqrt{t}) \\
\mu^2
\end{pmatrix},
\begin{pmatrix}
\mp\mu(\lambda^2 + 1)/(\lambda \sqrt{t}) \\
-2\mu^2 \\
0
\end{pmatrix},
\begin{pmatrix}
2/\sqrt{t} \\
\pm\mu(\lambda^4 - \lambda^2 - 2)/(\lambda \sqrt{t}) \\
3\mu^2
\end{pmatrix} \rangle
\]

Scalar multiples \(u_{31}, u_{32}, u_{33}\) (\(u_{11}, u_{22}, u_{23}\) in trivial case) do not influence to determine the each 3-subspaces. More precisely, these scalars arise from the isotropy group \(Iso(\beta)\). Indeed the dimension of \(Iso(\beta)\) is three in \(GL(3)\), or two in \(SL(3)\). The next property is the condition of \(U \in SL(3)\). In this paper, this fact essentially is not used, but I’ll mention. Since \(\det U = 1\), we get

\[
u^3 = \lambda/(2(\lambda^2 + 1)^2\mu) (< 0).
\]

We’ll show the transversality in each intersections. It may be assumed that the 3-subspace \(F_{t-1}(\alpha_t)\) is generated by 2-subspace \(\beta\) and the following symmetric matrix \(B^t\):

\[
B^t = \begin{bmatrix}
b_1 & 0 & 0 \\
0 & b_2 & b_3 \\
0 & b_2 & b_3
\end{bmatrix}
\]

If the intersection is trivial then \(U = \text{identity matrix} \ I\) and \(b_1 = 1, b_2 = b_4 = 0\) and \(b_3 = -t\). And so we’ll considered the case \(U \neq I\).

**Lemma** If we set \(u_{32}/u_{31} = (-\lambda^2 + 7)/2u_{33},\) then \(b_1 = 1, b_3 = s/k_1, b_2 = b_4 = 0,\) where \(s = u_{31}/u_{33}, k_1 = -(\lambda^2 + 6)\).

**Proof of lemma** Let \(U = (v_1, v_2, v_3)\). First, we’ll determine the subspace \(F_{tU}(\alpha_t^+)\), this is achieved by calculating the elements \(tP(v_1) + P(v_3), Q(v_2, v_3), Q(v_1, v_3) + P(v_2)\). Using the expression (12), we have followings:

\[
tP(v_1) + P(v_3) = (\lambda^2 + 1) \begin{bmatrix}
0 & 0 & u_{32}^2 \\
0 & u_{32}^2 & u_{32}(u_{33} - \lambda^2 u_{32}/u_{31}) \\
u_{32}^2 & u_{32}(u_{33} - \lambda^2 u_{32}/u_{31}) & (u_{33} - \lambda^2 u_{32}/u_{31})^2 + \lambda^2 u_{32}^4/u_{31}^2
\end{bmatrix}
\]

\[
Q(v_2, v_3) = \pm 2\mu/\sqrt{t} \begin{bmatrix}
-u_{31}^2 & 0 & u_{32}^2 \\
0 & u_{32}^2 & u_{32}(u_{33} + u_{32}/u_{31}) \\
u_{32}^2 & u_{32}(u_{33} + u_{32}/u_{31}) & u_{31}^2 + 2u_{32}^2 u_{33}/u_{31}
\end{bmatrix}
\]
Then we may only examine manifold $G$ (Q.E.D of the lemma
From $\text{cl}$ span orb with tangent space of $G$ + matrices to $F$ we put the orbit of $\beta$
whose conclusion induced by the assertion (Q.E.D)

Then if $a$ $(\beta, \delta)$, we may only understand that the tangent space at $\beta$ of $\text{orb}(\beta)$ is spanned by $\text{hom}(\beta, \delta)$ modulo to the subspace $F_{U}^{-1}(\alpha_{i})$. We'll recall the classification theorem of $\text{cl}(\text{orb}(\beta))$, the Lie algebra $\mathfrak{g}(\beta)$ of the isotropy group of $\beta$ is following:

$\mathfrak{g}(\beta) = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & -a_{11} \end{pmatrix}$

Then we may only examine $f_{a}$ for under the matrix $a$.

$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & -2a_{11} & 0 \\ a_{31} & a_{32} & a_{11} \end{pmatrix}$

$f_{a} (x_{1}Q_{12} + x_{2}(Q_{13} − 2P_{2})$ modulo to $F_{U}^{-1}(\alpha_{i})$ is;

$$
\begin{pmatrix}
0 & 0 & 0 \\
2(a_{21} + a_{32} - a_{12}b_{2}) & a_{31} & 0 \\
0 & 2a_{12}b_{3} & a_{11}
\end{pmatrix}
\begin{pmatrix}
x_{1} \\
x_{2}
\end{pmatrix}
+ 
\begin{pmatrix}
0 & 0 & 0 \\
2(6a_{11} - a_{13}b_{2}) & a_{21} - 2a_{32} & 0 \\
0 & 2(a_{31} - a_{13}b_{3}) & 2a_{21} - 2a_{32}
\end{pmatrix}
\begin{pmatrix}
x_{1} \\
x_{2}
\end{pmatrix}
$$

Then if $b_{3} \neq 0$, this span the $\text{hom}(\beta, \delta)$. This is completed the assertion.
(Q.E.D)

3 A conclusion induced by the assertion

We put the orbit of $\beta = \langle Q_{12}, Q_{13} − 2P_{2} \rangle$ by the group of nonsingular upper triangular matrices to $\{ \beta(a_{1}, a_{2}, l) | a_{1}, a_{2}, l > 0 \}$, where

$$
\beta(a_{1}, a_{2}, l) = \langle a_{1}P_{1} + Q_{12}, a_{2}P_{1} + t(P_{1} - P_{2}) + Q_{13} \rangle.
$$
When \( a_1 = a_2 = 0 \), for simplification of the notation we denote \( \beta(0,0,t) \) to \( \beta_t \). Easily the followings will be understood (besides we remark that the same properties is ture for non zero \( t \)):

\[
\text{orb}(\beta) = \bigcup \{ \text{SO}(3) \text{ orbit of } \beta(a_1,a_2,t) \mid \text{any } a_1, a_2, \ t > 0 \},
\]

and

\[
\text{orb}(\beta) \cap G_2(\langle I \rangle^+) = \bigcup_{t>0} \{ \text{SO}(3) \text{ orbit of } \beta_t \}.
\]

**Proposition 3.1** If a symmetric matrix \( A \) in \( \beta_t \) has the multiple eigen values, then \( A = (1/\sqrt{2})(P_1 - P_2) + Q_{13} \) up to scalar multiple.

**Proof** The characteristic equation of the problem is following:

\[
\begin{vmatrix}
  t - \lambda & x & 1 \\
  x & -t - \lambda & 0 \\
  1 & 0 & -\lambda
\end{vmatrix} = 0
\]

where any \( x, t > 0 \). The condition of multiple eigenvalues is \((x^2 + t^2 + 1)/3 - \sqrt{t^2/4} = 0\). This expression is deform to \( x^2/3 + (1/3)\sqrt{t^2/4} + 1)(2\sqrt{t^2/4} - 1)^2 = 0 \). Then we get the conclusion: i.e. \( x = 0, \ t^2 = 1/2 \).

(Q.E.D)

**Proposition 3.2** If a symmetric matrix \( A \) in \( \beta(a_1,a_2,t) \) has trace zero and the multiple eigen values, then \( a_2 = 0 \) and \( A = (1/\sqrt{2})(P_1 - P_2) + Q_{13} \) up to scalar multiple.

**Proof** When \( a_1 \neq 0 \), from the trace condition of \( A \) we get \( A = -(a_2/a_1)Q_{12} + t(P_1 - P_2) + Q_{13} \). From the proposition 3.1 we take \( a_2 = 0 \) and \( t = 1/\sqrt{2} \). The otherwise when \( a_1 = 0 \), from the trace condition of \( A \) we get \( a_2 = 0 \) too.

(Q.E.D)

Let \( \Sigma = \{ \alpha \mid P_1 + P_2 - 2P_3 \in \alpha, \alpha \in \text{orb}(\beta) \} \). Then there is some \( U \in \text{SO}(3) \) and some \( A \in \beta(a_1,a_2,t) \), we have \( P_1 + P_2 - 2P_3 = FUA \). Applying upper Propositions to \( A \), we have \( a_2 = 0 \) and \( t = 1/\sqrt{2} \). All elements \((u_1,u_2,u_3) \in \text{SO}(3)\) preserve the matrix \( P_1 + P_2 - 2P_3 \) are of a subgroup of \( \text{SO}(3) \), this subgroup is isomorphic to \( \text{SO}(2) \). Let \( \bar{\Sigma} = \{ \alpha \mid P_1 + P_2 - 2P_3 \in \alpha, \alpha \in \text{cl}(\text{orb}(\beta)) \} \). If \( \alpha \in \text{cl}(\text{orb}(\beta)) - \text{orb}(\beta) \), then from the structure theorem of \( \text{cl}(\text{orb}(\beta)) \), we obtain

\[
\alpha = \langle P(u_1), a_1Q(u_1,u_2) + t(P(u_1) - P(u_2)) + Q(u_1,u_3) \rangle \quad (t > 0)
\]

for \((u_1,u_2,u_3) \in \text{SO}(3)\). Applying the proposition 3.1 to \( \alpha \in \bar{\Sigma} \), we obtain \( a_1 = 0 \) and \( t = 1/\sqrt{2} \). If \( \gamma \in \bar{\Sigma} - \Sigma \) then \( \gamma = FU \alpha \) for \( \alpha = \langle P(u_1), P_1 + P_2 - 2P_3 \rangle \), where \( U = (u_1,u_2,u_3) \in \text{SO}(3) \) and \( U \) satisfies the following condition:

\[
P_1 + P_2 - 2P_3 = -\sqrt{2}(1/\sqrt{2})(P(u_1) - P(u_2) + Q(u_1,u_3)).
\]

Therefore any element \( \alpha \) of \( \bar{\Sigma} \) represent as \( \langle P(u_1)\sin \phi + Q(u_1,u_2)\cos \phi, P_1 + P_2 - 2P_3 \rangle \) where \(|\phi| \leq \pi/2\).
Proposition 3.3  \( \tilde{\Sigma} \) is diffeomorphic to 2-torus.

Proof  Let

\[
V = \begin{pmatrix}
1/\sqrt{3} & 0 & -\sqrt{2}/3 \\
0 & 1 & 0 \\
\sqrt{2}/3 & 0 & 1/\sqrt{3}
\end{pmatrix}.
\]

Then the orthogonal matrix \( V \) satisfies the condition : \( F_V( (1/\sqrt{2})(P_1 - P_2) + Q_{13}) = -(1/\sqrt{2})(P_1 + P_2 - 2P_3) \). Let \( A = F_V(P_1 \sin \phi + Q_{12} \cos \phi) \) for \(|\phi| \leq \pi/2\). Let \( W = (w_1, w_2, e_3) \) be a orthogonal matrix. It is saisfied that \( F_W(P_1 + P_2 - 2P_3) = P_1 + P_2 - 2P_3 \) by each \( W = (w_1, w_2, e_3) \). The set consist of this \( W \) is homomorphic to \( SO(2) \) and is parametrized by \( \theta(mod.2\pi) \). Let any \( \gamma \in \tilde{\Sigma} \) be represented by \( \langle F_WA, P_1 + P_2 - 2P_3 \rangle \). We can check \( (\phi(mod.\pi), \theta(mod.\pi)) \) is a chart of \( \tilde{\Sigma} \).

(Q.E.D)

Definition  A linear transformation between symmetric matrices \( \rho : S(3) \rightarrow \langle P_3 \rangle^+ \) is defined by \( A \mapsto A + (1/2)(trAP_3)(P_1 + P_2 - 2P_3) \). The linear transformation \( \rho \) induces the map from \( cl(\text{orb}(\beta)) - \tilde{\Sigma} \) to \( G_2(\langle P_3 \rangle^+) \). For this induced map, we use the same symbol \( \rho \) whenever without the confusion.

Proposition 3.4  The map \( \rho \) from \( cl(\text{orb}(\beta)) - \tilde{\Sigma} \) to \( G_2(\langle P_3 \rangle^+) \) is extendable to the map \( \tilde{\rho} \) on \( \sigma_{\Sigma}(cl(\text{orb}(\beta))) \).

Proof  Let \( v(\Sigma) \) be the normal bundle of 2-submanifold \( \Sigma \) in 6-manifold \( \text{orb}(\beta) \). From the structure theorem, One dimentional manifold \( \tilde{\Sigma} - \Sigma \) is contained within a stratum of the stratified set \( cl(\text{orb}(\beta)) \). Let \( v(\Sigma - \Sigma) \) be the normal bundle of \( \tilde{\Sigma} - \Sigma \) in this stratum which is a 5-manifold. Then let \( i \) be the projection of total space \( E(v(\Sigma)) \) to base space \( \tilde{\Sigma} \). We consider the blowing up \( cl(\text{orb}(\beta)) \) at each points of \( \tilde{\Sigma} \) that we denote to \( \sigma_{\Sigma}(cl(\text{orb}(\beta))) \). To make this process we must take a coordinate neighbourhood of each \( \alpha \in \tilde{\Sigma} \), For any \( \alpha \in \tilde{\Sigma} \), we can put \( \alpha = (P(u_1) \sin \phi + Q(u_1, u_2) \cos \phi, P_1 + P_2 - 2P_3) \), where

\[
P_1 + P_2 - 2P_3 = -\sqrt{2}(1/\sqrt{2}(P(u_1) - P(u_2)) + Q(u_1, u_3))
\]

and \( U = (u_1, u_2, u_3) \in SO(3) \).

The fiber \( i^{-1}(\alpha) \) on \( \alpha \) is generated by the derivative of the following 2-subspace \( \alpha_i \in \text{orb}(\beta) \). Let \( \alpha_i = (A_i, B_i) \), where \( A_i, B_i \) are as follows :

\[
A_i = P(X_iu_1) \sin \phi + Q(X_iu_1, X_iu_2) \cos \phi,
B_i = tx_1(P(X_iu_1) \cos \phi - Q(X_iu_1, X_iu_2) \sin \phi) + tx_2P(X_iu_2) + (1/\sqrt{2})(P(X_iu_1) - P(X_iu_2)) + Q(X_iu_1, X_iu_3),
\]

where \( X_i = \exp(IX) \), \( X \notin g(\alpha_0) \) and

\[
X = \begin{pmatrix}
0 & 0 & x_3 \\
0 & 0 & x_4 \\
-x_3 & -x_4 & 0
\end{pmatrix}.
\]
We remark \( \alpha = \alpha_0 \), and \( \rho(B_0) = 0 \). Besides, \( \rho(A_i) = \langle \rho(A_i) \rangle, \rho(B_i) = \langle \rho(A_i) \rangle, \rho((B_i - B_0)/t) \rangle \). We define \( \tilde{\rho}(\alpha_0) = \lim_{t \to 0} \rho(\alpha_1) \).

Therefore \( \tilde{\rho}(\alpha_0) = \langle \rho(P(u_1) \sin \phi + Q(u_1, u_2) \cos \phi) \rangle, \rho(d/dt|_{t=0}B) \rangle \). The derivative \( d/de \|_{t=0}B \) is homogeneous with the variables \( x_i (i = 1, 2, 3, 4) \). This map can be considered from \( \sigma_\alpha(cl(orb(\beta))) \) which blows up \( cl(orb(\beta)) \) at \( \alpha \) along the fiber \( \iota^{-1}(\alpha) \), to \( G_2(\langle P_3 \rangle ^+) \).

Continuing this process at each point of \( \tilde{\Sigma} \), we obtain the extended continuous map \( \tilde{\rho} \) of \( \sigma_\Xi(cl(orb(\beta))) \) to \( G_2(\langle P_3 \rangle ^+) \). The subset \( \sigma_\Xi(orb(\beta)) \) is a open dense differential manifold in \( \sigma_\Xi(cl(orb(\beta))) \). The restriction of \( \tilde{\rho} \) on the differential manifold \( \sigma_\Xi(orb(\beta)) \) is a differential extention of \( \rho \).

(Q.E.D)

**Proposition 3.5** \( \sigma_\Xi(orb(\beta)) \) is a open dense manifold in \( \sigma_\Xi(cl(orb(\beta))) \). The restriction of map \( \tilde{\rho} \) on \( \sigma_\Xi(orb(\beta)) \) is a degree one to modulo 2.

**Proof** We may identify \( \sigma_\Xi(cl(orb(\beta)) = (cl(orb(\beta)) - \tilde{\Sigma}) \cup (RP(\nu_0(\tilde{\Sigma}))) \), where \( \nu(\tilde{\Sigma}) = \nu(\nu_0(\tilde{\Sigma})) \), \( RP(\tilde{\Sigma}) \) is the projective space bundle of vector bundle \( E \) without 0-section. For any \( \alpha \in G_2(\langle P_3 \rangle ^+) \) - \( \tilde{\rho}(RP(\nu_0(\tilde{\Sigma}))) \), we know \( \tilde{\rho}^{-1}(\alpha) \in G_2(\langle P_1 + P_2 - 2P_3, \alpha \rangle) \cap cl(orb(\beta)) \). Applying the assertion to \( G_2(\langle P_1 + P_2 - 2P_3, \alpha \rangle) \cap cl(orb(\beta)) \), we obtain that \( \tilde{\rho} \) is surjective. \( \sigma_\Xi(orb(\beta)) \) is a open dense manifold in \( \sigma_\Xi(cl(orb(\beta))) \). The restriction of \( \tilde{\rho} \) on the open manifold \( \sigma_\Xi(orb(\beta)) \) is differential. We use the Sard's theorem to recognize the following fact. The \( \tilde{\rho}^{-1}(\alpha) \) is transversally intersect \( orb(\beta) \) for almost all \( \alpha \). The assertion say that this intersection number is one to modulus 2 too. Therefore we get the conclusion.

(Q.E.D)

**Proposition 3.6** The map \( \tilde{\rho} \) is homotopic to collapsing map \( \tau \) of \( \sigma_\Xi(cl(orb(\beta))) \) to \( cl(orb(\beta)) \).

**Proof** The proof is simple. We may consider \( (1 - t)\tilde{\rho} + t\tau \) for the homotpy. If It is concluded that \( ((1 - t)\tilde{\rho} + t\tau)A \neq 0 \) for any symmetric matrix \( A(\neq 0) \in \sigma(\alpha) (\in cl(orb(\beta)) - \tilde{\Sigma}) \), then this map is well defined. \( ((1 - t)\tilde{\rho} + t\tau)A = A + ((1 - t)/2)tr(P_3A)(P_1 + P_2 - 2P_3) \) gives \( (1 - t)\tilde{\rho} + t\tau \) is transversal to \( cl(orb(\beta)) \).

(Q.E.D)

We easily obtain the following theorem, from a series of upper propositions.

**Theorem** The homology class \([cl(orb(\beta))]\) is homologous to the fundamental class \([G_2(\langle P_3 \rangle ^+)\)] in \( H_0(G_2(S(3)) : Z_2) \).

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