The investigation of the generalized functions via the heat kernel method

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Abstract. The aim of this report is to introduce the theory of the generalized functions via the heat kernel method which is our main investigation. Especially, we give the proof of the Schwartz kernel theorem for the tempered distributions via the heat kernel method.
Keyword: Generalized functions, Schwartz kernel theorem, Tempered distributions, Heat kernel method, Heat equation.

1 Introduction

We have mainly investigated the generalized functions, especially the dual space of the Gel'fand-Shilov spaces ([6]), via the heat kernel method. The heat kernel method, introduced by T. Matsuzawa in [10], is the method to characterize the generalized functions on the Euclidean space by the initial value of the solutions of the heat equation.

$$\boxed{\text{Generalized Functions}} \quad \underset{t\to 0+}{\overset{u*E}{\rightleftarrows}} \quad \boxed{\text{Solutions of Heat Equation with some estimate}}$$

$$u \quad \underset{t\to 0+}{\overset{u*E}{\rightleftarrows}} \quad U(x,t),$$

where $E(x,t)$ is the heat kernel on the Euclidean space defined by $E(x,t) = (1/\sqrt{4\pi t})^d e^{-\frac{|x|^2}{4t}}$.

By means of the heat kernel method, we easily consider the theory of the generalized functions. Therefore we can give the easier proof of some fundamental theorems for the generalized functions. For example, the following results is known: • The Schwartz kernel theorem by the heat kernel method ([3], [12]) • The Paley-Wiener theorem by the heat kernel method ([9], [19]) • The Edge-of-the-Wedge theorem by the heat kernel method ([18]) • The Bochner-Schwartz theorem by the heat kernel method ([4]) • The propagation of micro analyticity of positive definite functions ([20]) • The Asymptotic expansions of the solutions to the heat equation by the heat kernel method ([13], [14], [21], [22]).

In this report, to introduce our investigation concretely, we focus on the space of the tempered distributions which is a subspace in the dual space of a Gel'fand-Shilov space and we will give the proof of the Schwartz kernel theorem for the tempered distributions via the heat kernel method. Moreover we also introduce the Schwartz kernel theorem for the tempered distributions on the Heisenberg group via the heat kernel method as our recent result.

The Schwartz kernel theorem is not only a fundamental theorem in the theory of the generalized functions but also am important result to consider the property of the Weyl transform as the operator (see [12]) and to consider BIBO (Bounded-Input Bounded-Output) stable for the LTI system theory (the linear time-invariant system theory) (see [1] and [15]).

2 $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$

We define the space $\mathcal{S}(\mathbb{R}^d)$ and its dual space $\mathcal{S}'(\mathbb{R}^d)$.

• $\mathcal{S}(\mathbb{R}^d) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \forall\alpha,\ \beta \in \mathbb{Z}_+^d,$
$\|\varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta \varphi(x)| < \infty\}.$

The space $\mathcal{S}(\mathbb{R}^d)$ is called the Schwartz class.

• We say T is in the space of the tempered distributions and denote by $T \in \mathcal{S}'(\mathbb{R}^d)$ if T is a linear map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathbb{C}$ and satisfies the following estimate:

$$|\langle T, \varphi\rangle| \le C\|\varphi\|_{\alpha,\beta},\ \forall\varphi \in \mathcal{S}(\mathbb{R}^d)$$

for some constant $C > 0$ and some $\alpha, \beta \in \mathbb{Z}_+^d$.

Example 1. **1.** $e^{-x^2} \in \mathcal{S}(\mathbb{R})$,

2. $\dfrac{1}{\cosh x} \in \mathcal{S}(\mathbb{R})$,

3. $E(x,t) \in \mathcal{S}(\mathbb{R}_x^d)$,

4. $\delta \in \mathcal{S}'(\mathbb{R})$, where δ is the Dirac's delta function,

5. $H(x) = \begin{cases} 1, & (x \ge 0), \\ 0, & (x < 0) \end{cases} \in \mathcal{S}'(\mathbb{R})$,

6. $P(x) \in \mathcal{S}'(\mathbb{R})$, where $P(x)$ is a polynomial,

7. $e^x \notin \mathcal{S}'(\mathbb{R})$.

3 The heat kernel method for $\mathcal{S}'(\mathbb{R}^d)$

The following Theorem 1 is the heat kernel method for the space $\mathcal{S}'(\mathbb{R}^d)$:

Theorem 1 ([11])**.** *Let $u \in \mathcal{S}'(\mathbb{R}^d)$. If we put $U(x,t) = \langle u, E(x-\cdot,t)\rangle$, then we have*

1. $U(x,t) \in C^\infty(\mathbb{R}^d \times (0,\infty))$,

2. $\left(\frac{\partial}{\partial t} - \Delta_x\right) U(x,t) = 0,\ (x,t) \in \mathbb{R}^d \times (0,\infty),\ \Delta_x = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$,

3. *for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$,*

$$\lim_{t\to+0} \int_{\mathbb{R}^d} U(x,t)\varphi(x)dx = \langle u, \varphi\rangle,$$

4. *there exist a constant $C > 0$ and $\mu, \nu \geq 0$ such that*

$$|U(x,t)| \leq Ct^{-\mu}(1+|x|)^\nu,\ x \in \mathbb{R}^d,\ 0 < t < 1.$$

Conversely, if the C^∞-function $U(x,t)$ satisfies the condition 2 and 4, there exists $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $U(x,t) = \langle u, E(x-\cdot,t)\rangle$.

Remark 1. By Theorem 1, we can see that for the solution of the heat equation $U(x,t)$ with the estimate in the condition 4, there exists $u \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\lim_{t\to+0} \int_{\mathbb{R}^d} U(x,t)\varphi(x)dx = \langle u, \varphi\rangle$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$. We use this fact in the proof of the Schwartz kernel theorem (Theorem 2) for the space $\mathcal{S}'(\mathbb{R}^d)$.

Remark 2. Several mathematicians show the heat kernel method for some functional spaces as follows:

Heat Kernel Methods
$\mathcal{A}'(K)$ (T. Matsuzawa, 1987, [10])
$\mathcal{S}'$ (T. Matsuzawa, 1990, [11])
$(\mathcal{S}_1^1)'$ (Korean Group, 1993, [8])
$\mathcal{G}'$ (Korean Group, 1994, [2])
$(\mathcal{S}_r^r)'$ (C. Dong and T. Matsuzawa, 1994, [5])
$(\mathcal{S}_1)'$ (M. Suwa, 2004, [18])
$(\mathcal{S}_r)'$ (Y. Oka and K. Yoshino, [14])
$(\mathcal{S}_r)'(A)$ (Y. Oka and K. Yoshino, [14])

• $A'(K)$: the space of the hyperfunctions with a compact support K, • $(\mathcal{S}_1^1)'(\mathbb{R}^d)$: the space of the Fourier hyperfunctions, • $\mathcal{G}'(\mathbb{R}^d)$: a space of the Fourier ultrahyperfunctions, • $(\mathcal{S}_1)'(\mathbb{R}^d)$: a space of the distributions with exponential growth, • $(\mathcal{S}_r)'(\mathbb{R}^d)$: the dual space of the Gel'fand-Shilov space $\mathcal{S}_r(\mathbb{R}^d)$, • $(\mathcal{S}_r)'(A)$: the the dual space of the Gel'fand-Shilov space $\mathcal{S}_r(\mathbb{R}^d)$ with regular closed support A.

4 The Schwartz kernel theorem for $\mathcal{S}'(\mathbb{R}^d)$

We will give the new proof of the following Schwartz kernel theorem for the space $\mathcal{S}'(\mathbb{R}^d)$ via the heat kernel method.

Theorem 2 ([16])**.** *Let k be a continuous and linear map from $\mathcal{S}(\mathbb{R}^{d_2})$ to $\mathcal{S}'(\mathbb{R}^{d_1})$.*
Then there exists $K \in \mathcal{S}'(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ such that

$$\langle k\varphi, \psi\rangle = \langle K, \varphi \otimes \psi\rangle = \iint K(x,y)\varphi(x)\psi(y)dxdy$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^{d_1})$ and $\psi \in \mathcal{S}(\mathbb{R}^{d_2})$.

Proof. Since k is continuous, the bilinear form $\mathbb{B}$ on $\mathcal{S}(\mathbb{R}^{d_1}) \times \mathcal{S}(\mathbb{R}^{d_2})$,

$$\mathbb{B}(\varphi, \psi) = \langle k\psi, \varphi\rangle\ ,\ \varphi \in \mathcal{S}(\mathbb{R}^{d_1})\ ,\ \psi \in \mathcal{S}(\mathbb{R}^{d_2})$$

is separately continuous. Since $\mathcal{S}(\mathbb{R}^{d_1})$ and $\mathcal{S}(\mathbb{R}^{d_2})$ is Fréchet space, $\mathbb{B}$ is continuous. Hence, there exist a constant $C > 0$ and $\alpha, \beta, \alpha', \beta' \in \mathbb{Z}_+^d$ such that

$$|\langle k\psi, \varphi\rangle| \leq C\|\varphi\|_{\alpha,\beta}\|\psi\|_{\alpha',\beta'}. \qquad (\sharp)$$

Set for $(x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $t > 0$,

$$K_t(x_1, x_2) = \langle kE(x_2 - \cdot, t),\ E(x_1 - \cdot, t)\rangle\ .$$

Now we show that K_t converges in $\mathcal{S}'(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ as $t\to+0$. By ($\sharp$), there exist a constant $C>0$ and $\mu_1,\mu_2,N_1,N_2\geq 0$ such that

$$|K_t(x_1,x_2)|\leq Ct^{-(\mu_1+\mu_2)}(1+|x_1|)^{N_1}(1+|x_2|)^{N_2}.$$

Moreover we can see

$$\left(\frac{\partial}{\partial t}-\Delta\right)K_t(x_1,x_2)=0$$

for $x_j\in\mathbb{R}^{d_j}$, $j=1,2$ and $t>0$.

Therefore, by Theorem 1, there exists $K_0\in\mathcal{S}'(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ such that $K_0=\lim\limits_{t\to+0}K_t$ in $\mathcal{S}'(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$.

For $\varphi\in\mathcal{S}(\mathbb{R}^{d_1})$, $\psi\in\mathcal{S}(\mathbb{R}^{d_2})$, we have

$$\begin{aligned}\langle K_t\ ,\ \varphi\otimes\psi\rangle&\\ =\iint_{\mathbb{R}^{d_1+d_2}}&K_t(x_1,x_2)\varphi(x_1)\psi(x_2)dx_1dx_2\\ =\iint_{\mathbb{R}^{d_1+d_2}}&\langle kE(x_2-y_2,t)\psi(x_2),\ E(x_1-y_1,t)\varphi(x_1)\rangle\,dx_1dx_2.\end{aligned}$$

Since the Riemann sum of an integral converges in $\mathcal{S}(\mathbb{R}^{d_j})$, $j=1,2$, we obtain

$$\begin{aligned}&\langle K_t\ ,\ \varphi\otimes\psi\rangle\\ &=\left\langle k\int_{\mathbb{R}^{d_2}}E(x_2-y_2,t)\psi(x_2)dx_2\ ,\ \int_{\mathbb{R}^{d_1}}E(x_1-y_1,t)\varphi(x_1)dx_1\right\rangle.\end{aligned}$$

Therefore we obtain $\langle K_0,\varphi\otimes\psi\rangle=\langle k\psi,\varphi\rangle$, as $t\to+0$. $\square$

5 Recent result

In [15], we obtain the Schwartz kernel theorem for the tempered distributions on the Heisenberg group, which is the 'most commutative' among the noncommutative Lie groups, via the heat kernel method. In this paper, we introduce the Heisenberg group, the heat kernel method for the tempered distributions on the Heisenberg group and the Schwartz kernel theorem for the tempered distributions on the Heisenberg group. In detail, we refer to [15].

5.1 The Heisenberg group $\mathbb{H}^d$

We recall the definition of the Heisenberg group (see [17]).

Let (x,y,t) and $(x',y',t')\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}=\mathbb{R}^{2d+1}$. Then we define the group law of $\mathbb{R}^{2d+1}$ by

$$\begin{aligned}&(x,y,t)(x',y',t')\\ &=(x+x',y+y',t+t'+2(x'\cdot y-x\cdot y')).\end{aligned}\tag{5.1}$$

The group $\mathbb{R}^{2d+1}$ with respect to the group law defined by (5.1) is called the Heisenberg group and denoted by $\mathbb{H}^d$. $\mathbb{H}^d$ is a locally compact Hausdorff group and its Haar measure is the Lebesgue measure $dxdydt$. The left-invariant vector fields on the Heisenberg group $\mathbb{H}^d$ as $\mathbb{R}^{2d+1}$ are represented by$X_j=\partial/\partial x_j+2y_j\partial/\partial t$, $X_{d+j}=\partial/\partial y_j-2x_j\partial/\partial t$ and $X_{2d+1}=\partial/\partial t$ for $j=1,2,\cdots,d$ and these make a basis for the Lie algebra of $\mathbb{H}^d$.

We denote by $\Delta_{\mathbb{H}^d}$ the sub-Laplacian on $\mathbb{H}^d$ defined by $\Delta_{\mathbb{H}^d}=\sum_{j=1}^{2d}X_j^2$ and consider the heat operator $\partial/\partial s-\Delta_{\mathbb{H}^d}$ on $\mathbb{H}^d\times(0,\infty)$. Let $\lambda>0$. Then we define the dilations δ_λ by $\delta_\lambda(x,y,t)=(\lambda x,\lambda y,\lambda^2 t)$ for $(x,y,t)\in\mathbb{H}^d$. Moreover, a function u from $\mathbb{H}^d$ to $\mathbb{C}$ is called the Heisenberg-homogeneous of degree $k\in\mathbb{Z}$ if $u\circ\delta_\lambda=\lambda^k u$ for $\lambda>0$. Especially the Heisenberg-homogeneous of degree of the distance function d defined by $d(x,y,t)=((x^2+y^2)^2+t^2)^{\frac14}$ for $(x,y,t)\in\mathbb{H}^d$ is one. The distance between two points (x,y,t) and (x',y',t') in $\mathbb{H}^d$ is given by $d\left((x',y',t')^{-1}(x,y,t)\right)$. Let f and h be suitable functions on $\mathbb{H}^d$. Then we define the convolution $f*h$ of f with h as follows: $(f*h)(x,y,t)=\int_{\mathbb{H}^d}f(x',y',t')h((x',y',t')^{-1}(x,y,t))dx'dy'dt'$ for $(x,y,t),(x',y',t')\in\mathbb{H}^d$.

For $\alpha\in\mathbb{Z}_+^{2d}$, the functions $(X_\alpha\varphi)(x,y,t)$ are defined by

$$(X_\alpha\varphi)(x,y,t)=(X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_{2d}^{\alpha_{2d}}\varphi)(x,y,t)$$

for a function $\varphi\in C^\infty(\mathbb{H}^d)$. The Schwartz class $\mathcal{S}(\mathbb{H}^d)$ on the Heisenberg group is defined by as follows:

Definition 1. For any $\varphi\in C^\infty(\mathbb{H}^d)$, we say $\varphi\in\mathcal{S}(\mathbb{H}^d)$ if the function φ satisfies the following condition: For any $N\in\mathbb{Z}_+$, we have

$$\begin{aligned}&\|\varphi\|_N\\ &=\sup_{(x,y,t)\in\mathbb{H}^d}(1+d(x,y,t))^N\sum_{|\alpha|\leq N}|X_\alpha\varphi(x,y,t)|<\infty.\end{aligned}$$

Moreover we denote by $\mathcal{S}'(\mathbb{H}^d)$ the dual space of $\mathcal{S}(\mathbb{H}^d)$ and we call it the space of the tempered distributions on the Heisenberg group.

Let $u\in\mathcal{S}'(\mathbb{H}^d)$ and $\varphi\in\mathcal{S}(\mathbb{H}^d)$. Then we define the convolutions $u*\varphi$ and $\varphi*u$ as follows:

$$\begin{aligned}\langle u*\varphi,\psi\rangle&=\langle u,\psi*\check{\varphi}\rangle\\ &=\int u(g)\left\{\int\varphi(g')\psi(gg')dg'\right\}dg,\ \ g,g'\in\mathbb{H}^d\end{aligned}$$

and

$$\begin{aligned}\langle\varphi*u,\psi\rangle&=\langle u,\check{\psi}*\varphi\rangle\\ &=\int u(g)\left\{\int\varphi(g')\psi(g'g)dg'\right\}dg,\ \ g,g'\in\mathbb{H}^d,\end{aligned}$$

where $\check{f}(g) = f(g^{-1})$ for $g \in \mathbb{H}^d$.

5.2 The heat kernel method for the space $\mathcal{S}'(\mathbb{H}^d)$

Theorem 3 ([7]). *For $u \in \mathcal{S}'(\mathbb{H}^d)$, we put $U_s(x,y,t) = (u * P_s)(x,y,t)$ for $(x,y,t) \in \mathbb{H}^d$ and $s > 0$. Then the function $U_s(x,y,t)$ satisfies the following four conditions: (1) $U_s(x,y,t) \in C^\infty(\mathbb{H}^d \times (0,\infty))$, (2) $(\partial/\partial s - \Delta_{\mathbb{H}^d})U_s(x,y,t) = 0$, $(x,y,t) \in \mathbb{H}^d$ and $s > 0$, (3) for any $\varphi \in \mathcal{S}(\mathbb{H}^d)$, $\langle u, \varphi \rangle = \lim_{s \to +0} \int_{\mathbb{H}^d} U_s(x,y,t)\varphi(x,y,t)dxdydt$, (4) there exist $\mu, \nu \geq 0$ and a constant $C > 0$ such that*

$$|U_s(x,y,t)| \leq Cs^{-\mu}(1 + d(x,y,t))^\nu$$

for $0 < s < 1$, $(x,y,t) \in \mathbb{H}^d$.

Conversely every $U_s(x,y,t) \in C^\infty(\mathbb{H}^d \times (0,\infty))$ satisfying the conditions (2) and (4) can be expressed in the form $U_s(x,y,t) = (u * P_s)(x,y,t)$ with the unique element $u \in \mathcal{S}'(\mathbb{H}^d)$, where P_s is the heat kernel associated to the sub-Laplacian $\Delta_{\mathbb{H}^d}$.

5.3 The Schwartz kernel theorem for the space $\mathcal{S}'(\mathbb{H}^d)$

By Theorem 3, we can prove the following Schwartz kernel theorem for the tempered distributions on $\mathbb{H}^d$ similarly as Theorem 2 (in detail, see [15]).

Theorem 4 ([15]). *Let k be a continuous linear operator from $\mathcal{S}(\mathbb{H}^{d_2})$ to $\mathcal{S}'(\mathbb{H}^{d_1})$. Then there exists T in $\mathcal{S}'(\mathbb{H}^{d_1} \times \mathbb{H}^{d_2})$ such that*

$$\langle k\psi, \varphi \rangle = \langle T, \varphi \otimes \psi \rangle = \iint T(g_1, g_2)\varphi(g_1)\psi(g_2)dg_1dg_2,$$

where φ is in $\mathcal{S}(\mathbb{H}^{d_1})$ and ψ is in $\mathcal{S}(\mathbb{H}^{d_2})$.

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