The parameter space and Hyper-surface regularization of Feynman integrals

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Abstract

We showed the basic mathematical features of our new parameter space and its applications to Feynman integrals in the previous paper last year. In this paper we correct and modify a few ambiguities and mistakes in my paper of twelve years ago and present a new regularization of Feynman integrals applying our parameter space and the integral method to the calculation of $K_S^0 \to 2\gamma$ decay width. Furthermore we discuss how to establish the integral domain in Feynman integral, comparing the results of our calculations with the data of $K_S^0 \to 2\gamma$ decay and then using another useful method of Feynman integral calculations, the so called Davydychev method.

Key Words : Feynman integrals, quantum chromodynamics, Dimensional Regularization, hypergeometric function

1 Introduction

In the previous paper we discussed the fundamental mathematical features of our new parameter space and their applications to Feynman integrals last year [1]. In section 2 we correct and modify a few ambiguities and mistakes in my paper of twelve years ago [2], and present a new regularization of Feynman integrals and we will call this regularization Hypersurface regularization, applying our parameter space and the integral method to the calculation of $K_S^0 \to 2\gamma$ decay width. In section 3 we try to calculate the decay width of the same process as section 2 by taking the different integral domain and compare the result of this calculation with the experimental data . We can understand that the result of the calculation in section 3 does not coincide with the experimental data, though the result in section 2 coincides with the data precisely. I think it is a very important result. In section 4 we use Davydychev method [5][6] to calculate the same Feynman integral and try to compare the result of the calculation with the results in section 2 and in section 3.

2 The application of this parameter transformation to $K_s^0 \to 2\gamma$ Decay

At first we review the topics of my paper published twelve years ago, and correct mistakes of the sign and the variables [2]. Further we explain the mathematical meanings of the integral method and the parametrization as a new regularization.

We calculate $\Gamma(K_s^0 \to 2\gamma)/\Gamma(K_s^0 \to \pi^+\pi^-)$ as the application of our new parameter transformation.
In Fig.1 we show the diagrams we have to calculate, where we neglect a possible diagram that an outgoing photon emits out of the bottom of the incoming particle K_S^0 [3].

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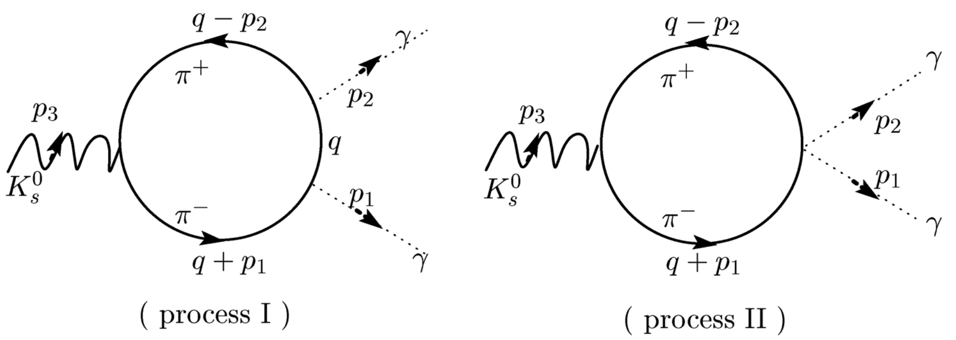

Fig.1

The amplitude is the following ,

$$< p_1, p_2|S|p_3 >= \delta^4(p_1+p_2-p_3)\frac{2ge^2\epsilon_\lambda^{(1)}\epsilon_\mu^{(2)}}{(2p_{10}V)^{1/2}(2p_{20}V)^{1/2}(2p_{30}V)^{1/2}} \times \Big[\int d^4q \frac{i(2q+p_1)_\lambda i(2q-p_2)_\mu}{[(q+p_1)^2-\mu^2+i\epsilon][q^2-\mu^2+i\epsilon][(q-p_2)^2-\mu^2+i\epsilon]} + \delta_{\mu\lambda}\int d^4q\frac{1}{[(q+p_1)^2-\mu^2+i\epsilon][(q-p_2)^2-\mu^2+i\epsilon]}\Big], \tag{1}$$

where p_i, q are momenta of particles and $\epsilon_\mu^{(i)}$ is the polarization vector of photon.

If we assume $(\epsilon^{(1)}q)(\epsilon^{(2)}q) = (\epsilon^{(1)}\epsilon^{(2)})\frac{1}{4}(q^2+F(s_1))$ and $(\epsilon^{(1)}p_1) = (\epsilon^{(2)}p_2) = 0$,
we have

$$< p_1, p_2|S|p_3 >= (2\pi)^4\delta^4(p_1+p_2-p_3)\frac{2ge^2(\epsilon^{(1)}\epsilon^{(2)})}{(2p_{10}V)^{1/2}(2p_{20}V)^{1/2}(2p_{30}V)^{1/2}}\Big[-\mu^2 I(q)+F(s_1)I(q)\Big], \tag{2}$$

where

$$I(q) = \frac{1}{(2\pi)^4}\int d^4q\frac{1}{[(q+p_1)^2-\mu^2+i\epsilon][q^2-\mu^2+i\epsilon][(q-p_2)^2-\mu^2+i\epsilon]}. \tag{3}$$

We used Lorentz condition since the real photons are transverse waves and they fulfill Lorentz condition. The function $F(s_1)$ is added in order that $(\epsilon^{(1)}q)(\epsilon^{(2)}q)$ isn't gauge invariant and it is a function of the momentum square of K_s^0 meson $p_3^2 = s_1$. We have to calculate this integral (3) in Euclidean space to use our new parameter transformation. Then we transform Euclidean metric into Bjorken-Drell metric. In Euclidean momentum space the equation (3) is expressed as follows:

$$I(q) = \frac{-i}{(2\pi)^4}\int\frac{d^4q}{[(q+p_1)^2+\mu^2][q^2+\mu^2][(q-p_2)^2+\mu^2]}. \tag{4}$$

We use our parameter transformation to the integral (4) in 2ω-dimensional momentum Euclidean space,

$$\begin{aligned} I(q) &= -i\prod_{i=1}^{3}\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\int_0^\infty dt_i \exp[-(q^2+\mu^2)t_1-((q+p_1)^2+\mu^2)t_2-((q-p_2)^2+\mu^2)t_3] \\ &= -i\int\frac{d^{2\omega}q}{2\pi^{2\omega}}\prod_{i=1}^{3}\int_0^\infty dt_i \\ &\quad \times\exp[-(\sum_{i=1}^{3}t_i)(q+\frac{p_1t_2-p_2t_3}{\sum_{i=1}^3 t_i})^2+\frac{(p_1t_2-p_2t_3)^2}{\sum_{i=1}^3 t_i}-(\sum_{i=1}^{3}t_i)\mu^2]. \end{aligned} \tag{5}$$

We shift the momentum q from q to $q' = q + \frac{p_1 t_2 - p_2 t_3}{\sum_{i=1}^{3} t_i}$,

$$I(q') = -i \int \frac{d^{2\omega} q'}{(2\pi)^{2\omega}} \prod_{i=1}^{3} \int_0^\infty dt_i \exp\Big[-(\sum_{i=1}^{3} t_i)(q')^2 + \frac{(p_1 t_2 - p_2 t_3)^2}{\sum_{i=1}^{3} t_i} - (\sum_{i=1}^{3} t_i)\mu^2 \Big]. \tag{6}$$

We perform the 2ω dimensional gaussian integral with respect to q'.
The result is

$$I(p_i) = -i \prod_{i=1}^{3} \int_0^\infty (\frac{1}{4\pi \sum_{i=1}^{3} t_i})^\omega dt_i \exp\Big[\frac{(p_1 t_2 - p_2 t_3)^2}{(\sum_{i=1}^{3} t_i)^2} - (\sum_{i=1}^{3} t_i)\mu^2 \Big]. \tag{7}$$

We adopt the parameter transformation and its Jacobian [2],

$$x_1 = \sum_{i=1}^{3} t_i, \;\; x_2 = t_2 / \sum_{i=1}^{3} t_i, \;\; x_3 = t_3 / \sum_{i=1}^{3} t_i \;\; \text{and} \;\; \frac{\partial(t_1, t_2, t_3)}{\partial(x_1, x_2, x_3)} = x_1^2. \tag{8}$$

Then the equation (7) becomes as follows:

$$I(p) = -i(4\pi)^{-\omega} \int_0^\infty dx_1 \int_0^1 dx_2 \int_0^1 dx_3 x_1^{2-\omega} \exp(-x_1 S(x,p)), \tag{9}$$

where
$S(x,p) = \mu^2 - 2p_1p_2x_2x_3 = \mu^2 - s_1x_2x_3$, and $s_1 = (p_1+p_2)^2 = 2p_1p_2 = p_3^2$ expresses Mandelstam variable, and $p_1^2 = p_2^2 = 0$ because the external photon masses are 0.
In this stage we replace Euclidean momentum space with Minkowski momentum space, that is $S(x,p) = -\mu^2 + 2p_1p_2x_2x_3 = -\mu^2 + s_1x_2x_3$.
If $S(x,p)$ is positive, the integral with respect to x_1 converges. Estimating μ^2 and s_1 at pion mass squared $\mu^2 = 0.0195\text{Gev}^2$ and kaon mass squared $s_1 = 0.248\text{Gev}^2$, $S(x,p)$ is positive in the region $0 \le x_2 \le 1$ and $0 \le x_3 \le 1$ as we can see that in Fig.2.

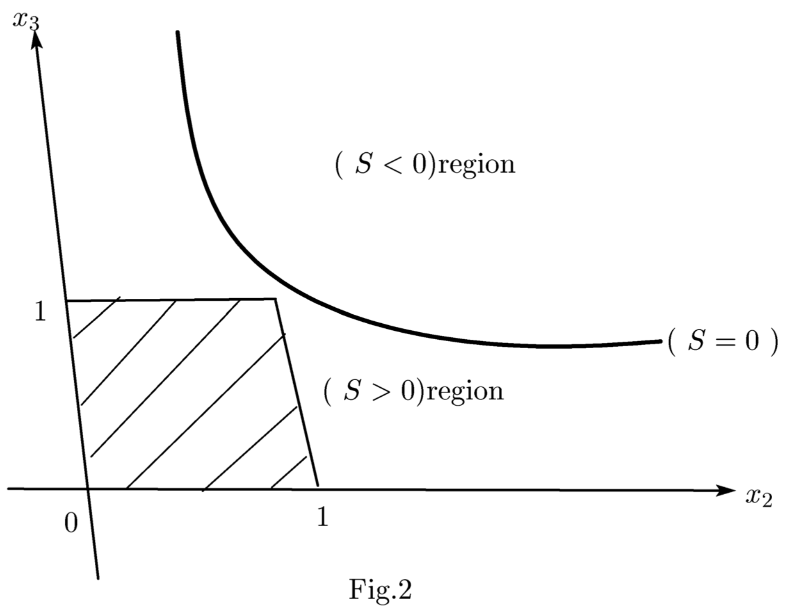

Fig.2

Changing the integral variable x_1 from x_1 to $y = S(x,p)x_1$, because $dx_1 = \frac{1}{S(x,p)}dy$, the equation

(9) becomes as follows:

$$I(p) = -i(4\pi)^{-\omega}\int_0^\infty dy\int_0^1 dx_2\int_0^1 dx_3 y^{2-\omega}\exp(-y)\frac{1}{S(x_2x_3,p)^{3-\omega}}. \tag{10}$$

Then we perform the gamma integral concerning y.
The result is

$$I(p) = -i(4\pi)^{-\omega}\Gamma(3-\omega)\int_0^1 dx_2\int_0^1 d_3 S(x,p)^{\omega-3} = -i(4\pi)^{-\omega}\tilde{I}(s_1). \tag{11}$$

When we take the limit $\omega \to 2$, because there are not any singularities with respect to ω in the integrand, we can put $\omega = 2$. This means analytic continuation from 2ω dimensional space to 4 dimensional Minkowski space. Performing the x_2, x_3 integration, the result is

$$\begin{aligned}\tilde{I}(s_1) &= -\frac{1}{s_1}\log|\mu^2|\log|\epsilon| + \frac{1}{s_1}\log|\mu^2|\log|\epsilon| - \frac{1}{s_1}L_{i2}(\frac{s_1}{\mu^2}) \ \ (\epsilon \to +0)\\ &= -\frac{1}{s_1}L_{i2}(\frac{s_1}{\mu^2}),\end{aligned} \tag{12}$$

where

$$L_{i2}(z) = -\int_0^z \frac{\log|1-u|}{u}\,du \tag{13}$$

is the so called Dilogarithm function. As you well know , in the first line of the equation (12) the first and the second terms are canceled out against each other and we can get rid of the divergence which causes from the lower limit of the integral . Furthermore we can represent the result of the integration in terms of gamma function and dilogarithm function.
The transition amplitude is defined as follows:

$$\begin{aligned}< p_1, p_2|S|p_3 > &= (2\pi)^4\delta^4(p_2+p_1-p_3)\frac{2ge^2}{(2p_{10}V)^{1/2}(2p_{20}V)^{1/2}(2p_{30}V)^{1/2}}\\ &\quad\times(-i)\frac{\mu^2}{s_1}(4\pi)^{-2}(\epsilon^{(1)}\epsilon^{(2)})\Big(L_{i2}(s_1/\mu^2) - \frac{1}{\mu^2}F(s_1)L_{i2}(s_1/\mu^2)\Big).\end{aligned} \tag{14}$$

Finally we have to calculate decay width $\Gamma(K_s^0 \to 2\gamma)$,

$$\Gamma(K_s^0 \to 2\gamma) = \frac{1}{2}\frac{V}{(2\pi)^4}\frac{V}{(2\pi)^3}\sum_{pol}\int d^3p_1\frac{V}{(2\pi)^3}\int d^3p_2\Big|< p_1, p_2|S|p_3 >\Big|^2. \tag{15}$$

Taking K_s^0 rest frame for simplicity, as final photons are massless,
the calculation $\int d^3p_1\int d^3p_2\delta^4(p_3-p_2-p_1)$ is trivial. We find

$$\Gamma(K_s^0 \to 2\gamma) = \frac{1}{16}\frac{g^2\alpha^2}{(2\pi)^3}\frac{\mu^4}{s_1^{5/2}}\Big\{L_{i2}(s_1/\mu^2) - \frac{1}{\mu^2}F(s_1)L_{i2}(s_1/\mu^2)\Big\}^2\sum_{pol}|(\epsilon^{(1)}\epsilon^{(2)})|^2. \tag{16}$$

We expand $L_{i2}(x)$ to Taylor Series,

$$L_{i2}\Big(\frac{s_1}{\mu^2}\Big) = \sum_{n=1}^\infty \frac{x^n}{n^2}\Big|_{x=s_1/\mu^2}. \tag{17}$$

We assume $\sum_{pol}\Big|\epsilon^{(1)}\cdot\epsilon^{(2)}\Big|^2 = 1/4$ and our term $\frac{1}{\mu^2}F(s_1)L_{i2}(s_1/\mu^2)$ to the following ,

$$\frac{1}{\mu^2}F(s_1)L_{i2}(s_1/\mu^2) = \sum_{n=3}^\infty \frac{x^n}{n^2}\Big|_{x=s_1/\mu^2}. \tag{18}$$

We can get the final result of the decay width $\Gamma(K_s^0 \to 2\gamma)$,

$$\Gamma(K_s^0 \to 2\gamma) = \frac{g^2\alpha^2\mu^4}{64(2\pi)^3 s_1^{5/2}}\Big(L_{i2}(s_1/\mu^2) - \frac{1}{\mu^2}F(s_1)L_{i2}(s_1/\mu^2)\Big)^2. \tag{19}$$

In Ref.[4], the decay width for $K_s^0 \to \pi^+\pi^-$ is given as

$$\Gamma(K_s^0 \to \pi^+\pi^-) = \Big(\frac{g^2}{4\pi}\Big)\frac{1}{4s_1^{1/2}}\Big(1 - \frac{4\mu^2}{s_1}\Big)^{1/2}. \tag{20}$$

Therefore we have the ratio $\Gamma(K_s^0 \to 2\gamma)/\Gamma(K_s^0 \to \pi^+\pi^-)$,

$$\frac{\Gamma(K_s^0 \to 2\gamma)}{\Gamma(K_s^0 \to \pi^+\pi^-)} = \frac{\alpha^2}{8\times(2\pi)^2}\frac{\Big(1+\frac{s_1}{4\mu^2}\Big)^2}{\Big(1-\frac{4\mu^2}{s_1}\Big)^{1/2}}. \tag{21}$$

We can estimate the numerical value of $\Gamma(K_s^0 \to 2\gamma)/\Gamma(K_s^0 \to \pi^+\pi^-)$
with
$\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$, $s_1 = 0.248\mathrm{GeV}^2$ (kaon mass squared), $\mu^2 = 0.0195\mathrm{GeV}^2$(pion mass squared) .
The result is

$$\frac{\Gamma(K_s^0 \to 2\gamma)}{\Gamma(K_s^0 \to \pi^+\pi^-)} = 3.56\times 10^{-6}. \tag{22}$$

This result is consistent with experimental data.

3 The calculation of Feynman integral in another integral domain

In this section we try to calculate Feynman integral in another integral domain. The integral domain isn't determined uniquely. We introduce the constraint $\sum_{i=2}^4 x_i = 1$ exactly in Feynman integral. The equation(9) is modified as follows:

$$I(p) = -i(4\pi)^{-\omega}\int_0^\infty dx_1\int_0^1 dx_2\int_0^1 dx_3\int_0^1 dx_4\delta\Big(1-\sum_{i=2}^4 x_i\Big)x_1^{2-\omega}\exp(-x_1 S(x,p)), \tag{23}$$

where $S(x,p) = -\mu^2 + 2p_1p_2x_2x_3 = -\mu^2 + s_1x_2x_3$, and $x_4 = t_1/(\sum_{i=1}^3 t_i)$. At first we calculate the integration concerning variables x_1 and x_4 . We can get $\Gamma(3-\omega)$ by integrating on the variable x_1 like section 2 . Because the integration on x_4 is the integration of Dirac's δ function , it is very easy. When ω moves in close to 2, because $\Gamma(3-\omega)$ doesn't have any singularities, we can do the analytic continuation from any ω to $\omega = 2$. Therefore $\Gamma(3-\omega) = \Gamma(1) = 1$. For short we omit the coefficients before the integration,

$$\tilde{I}(s_1) = \int_0^1 dx_2\int_0^{1-x_2}\frac{dx_3}{-\mu^2+s_1x_2x_3}. \tag{24}$$

Subsequently we have to calculate the integration on the variable x_3. The integral domain becomes $0 \le x_3 \le 1-x_2$ as the result of the integration of $\delta(1-\sum_{i=2}^4 x_i)$,

$$\begin{aligned}\tilde{I}(s_1) &= \int_0^1 dx_2\int_0^{1-x_2}\frac{dx_3}{-\mu^2+s_1x_2x_3} = \int_0^1 dx_2\Big[\frac{1}{s_1x_2}\log|\mu^2 - s_1x_2x_3|\Big]_0^{1-x_2}\\ &= \frac{1}{s_1}\int_0^1\frac{1}{x_2}\log|\mu^2 - s_1x_2 + s_1x_2^2|\,dx_2 - \frac{1}{s_1}\int_0^1\frac{1}{x_2}(\log\mu^2)\,dx_2\\ &= \frac{1}{s_1}\int_0^1\frac{dx_2}{x_2}\Big(\log s_1 + \log\Big|\frac{\mu^2}{s_1} - x_2 + x_2^2\Big|\Big) + \frac{1}{s_1}\log(\mu^2)\log\epsilon \quad (\epsilon\to 0).\end{aligned} \tag{25}$$

Putting

$$x_2^2 - x_2 + \frac{\mu^2}{s_1} = (x_2-\beta)(x_2-\gamma), \tag{26}$$

we obtain

$$\beta = \frac{1+\sqrt{1-\frac{4\mu^2}{s_1}}}{2}, \qquad \text{and} \qquad \gamma = \frac{1-\sqrt{1-\frac{4\mu^2}{s_1}}}{2}, \tag{27}$$

where

$$\beta+\gamma = 1, \qquad \text{and} \qquad \beta\gamma = \frac{\mu^2}{s_1}. \tag{28}$$

Substituting the equation (26) in the equation (25), we can get

$$\tilde{I}(\beta,\gamma) = \frac{1}{s_1}\int_0^1 \frac{1}{x_2}\log(\beta-x_2)\,dx_2 + \frac{1}{s_1}\int_0^1 \frac{1}{x_2}\log(\gamma-x_2)\,dx_2 + \frac{1}{s_1}\log\epsilon\log\Big|\frac{\mu^2}{s_1}\Big| \quad (\epsilon\to+0). \tag{29}$$

Furthermore we can express the equation (29) with Dilogarithm functions as follows:

$$\begin{aligned}\hat{I}(\beta) &= \frac{1}{s_1}\int_0^1 \frac{1}{x_2}\log(\beta-x_2)\,dx_2 = \frac{1}{s_1}\int_0^1 \frac{1}{x_2}\Big(\log\beta + \log\Big(1-\frac{x_2}{\beta}\Big)\Big)\,dx_2 \\ &= \frac{1}{s_1}\log\beta\Big[\log x_2\Big]_0^1 + \frac{1}{s_1}\int_0^1 \frac{\log\Big(1-\frac{x_2}{\beta}\Big)}{x_2}\,dx_2 = \frac{-1}{s_1}\log\beta\log(\epsilon) \\ &\quad + \frac{1}{s_1}\int_0^{\frac{1}{\beta}} \frac{\log(1-u)}{u}\,du = \frac{-1}{s_1}\log\beta\log(\epsilon) - \frac{1}{s_1}L_{i2}(1/\beta) \quad (\epsilon\to+0).\end{aligned} \tag{30}$$

And similarly

$$\hat{I}(\gamma) = \frac{1}{s_1}\int_0^1 \frac{1}{x_2}\log(\gamma-x_2)dx_2 = \frac{-1}{s_1}\log\gamma\log(\epsilon) - \frac{1}{s_1}L_{i2}(1/\gamma) \quad (\epsilon\to+0). \tag{31}$$

Putting the equation (30) and the equation (31) into the equation (29), we have

$$\begin{aligned}\tilde{I}(s_1) &= \hat{I}(\beta)+\hat{I}(\gamma)+\frac{1}{s_1}\log\epsilon\log\Big(\frac{\mu^2}{s_1}\Big) \\ &= \frac{1}{s_1}\log(\epsilon)\log\Big(\frac{\mu^2}{s_1}\Big) - \frac{1}{s_1}\log(\epsilon)\log\beta - \frac{1}{s_1}L_{i2}(1/\beta) - \frac{1}{s_1}\log(\epsilon)\log\gamma - \frac{1}{s_1}L_{i2}(1/\gamma) \\ &= \frac{1}{s_1}\log(\epsilon)\log\Big(\frac{\mu^2}{s_1\beta\gamma}\Big) - \frac{1}{s_1}\Big[L_{i2}(1/\beta)+L_{i2}(1/\gamma)\Big] \quad (\epsilon\to+0).\end{aligned} \tag{32}$$

Because $\beta\gamma = \mu^2/s_1$, so that $\log\Big(\frac{\mu^2}{s_1\beta\gamma}\Big) = \log 1 = 0$. In this way we can regularize Feynman integral here because the divergence term which occurs from the lower limit of the integral is eliminated and the result of the integral is expressed in the form of Dilogarithm function.
The final result is the following,

$$\tilde{I}(s_1) = -\frac{1}{s_1}\Big[L_{i2}(1/\beta)+L_{i2}(1/\gamma)\Big], \tag{33}$$

where $L_{i2}(1/\beta)$ and $L_{i2}(1/\gamma)$ are Dilogarithm functions defined as follows again:

$$L_{i2}(z) = -\int_0^z \frac{\log(1-u)}{u}\,du. \tag{34}$$

Finally we have to estimate the value of $\Gamma(K_S^0\to 2\gamma)/\Gamma(K_S^0\to\pi^+\pi^-)$ as the same as section 2 . We can obtain

$$\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta+\gamma}{\beta\gamma} = \frac{s_1}{\mu^2}, \quad \frac{1}{4}\Big(\frac{1}{\beta^2}+\frac{1}{\gamma^2}\Big) = \frac{1}{4}\Big[\frac{s_1^2}{\mu^4} - \frac{2s_1}{\mu^2}\Big] \tag{35}$$

from the equation (28). Furthermore taking

$$L_{i2}(1/\beta)+L_{i2}(1/\gamma)-\frac{1}{\mu^2}F(s_1)\Big[L_{i2}(1/\beta)+L_{i2}(1/\gamma)\Big]=\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{4\beta^2}+\frac{1}{4\gamma^2}=\frac{s_1}{2\mu^2}\Big(1+\frac{s_1}{2\mu^2}\Big), \quad (36)$$

and adopting $s_1=0.248\mathrm{GeV}^2$(Kaon mass squared) and $\mu^2=0.0195\mathrm{GeV}^2$(pion mass squared), the estimation of the value is the following,

$$\frac{\Gamma(K_s^0\to 2\gamma)}{\Gamma(K_s^0\to\pi^+\pi^-)}=\frac{\alpha^2}{8\times(2\pi)^2}\frac{\Big(\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{4\beta^2}\frac{1}{4\gamma^2}\Big)^2}{\Big(1-\frac{4\mu^2}{s_1}\Big)^{1/2}}=\frac{\alpha^2}{8\times(2\pi)^2}\frac{\frac{s_1^2}{4\mu^4}\Big(1+\frac{s_1}{2\mu^2}\Big)^2}{\Big(1-\frac{4\mu^2}{s_1}\Big)^{1/2}}=4.47\times10^{-4}. \quad (37)$$

It seems that this result is not consistent with experimental data. But more rigorous estimations will be needed.

4 The calculation of the Feynman integral by Davydychev method

This time we try to calculate Feynmann integral (3) using the Davydychev method [5][6]. This method is independent of the integral domains unlike the calculation in section 2 and section 3, although we can take several integral contours. Furthermore we can express the results of the calculations by using hypergeometric functions. Therefore we can do analytic continuation to the several domains and express the results with the convergent series easily,too.

At first from the equation (3) we have

$$J_3(1,1,1;u)=\int\frac{d^{2\omega}q}{[q^2-\mu^2+i\epsilon][(q+p_1)^2-\mu^2+i\epsilon][(q-p_2)^2-\mu^2+i\epsilon]}. \quad (38)$$

From now on we drop $i\epsilon$ for simplicity, namely we carry out the calculations on the pseudo Euclidean momentum space.
The equation (38) is

$$\begin{aligned}&J_3(1,1,1;\mu)=\\&=\int\frac{d^{2\omega}q}{[q^2-\mu^2][(q+p_1)^2-\mu^2][(q-p_2)^2-\mu^2]}=\int\frac{d^{2\omega}q}{q^2(1-\frac{\mu^2}{q^2})[(q+p_1)^2-\mu^2][(q-p_2)^2-\mu^2]}\\&=\int\frac{1}{q^2}{}_1F_0\Big(1;\frac{\mu^2}{q^2}\Big)\frac{d^{2\omega}q}{[(q+p_1)^2-\mu^2][(q-p_2)^2-\mu^2]},\end{aligned} \quad (39)$$

because $1/(1-\frac{\mu^2}{q^2})=\sum_{j=0}^{\infty}\frac{(1)_j}{j!}\Big(\frac{\mu^2}{q^2}\Big)^j$, where $(1)_j=1\times2\times\ldots\times j=j!$ from $(a)_j=(a)\times(a+1)\times\cdots\times(a+j-1)$.
Next we exploit Mellin Barnes integral representation of ${}_1F_0$ as follows:

$${}_1F_0\Big(1;\frac{\mu^2}{q^2}\Big)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(-v)\Gamma(1+v)\Big(-\frac{\mu^2}{q^2}\Big)^v dv. \quad (40)$$

The equation (39) becomes

$$J_3\Big(1,1,1;\mu\Big)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dv\Gamma(-v)\Gamma(1+v)(-\mu^2)^vJ_2(1+v,1,1;\mu), \quad (41)$$

where $J_2(1+v,1,1;\mu)$ is, referring the equation (29) in [1],

$$J_2(1+v,1,1;\mu)=\int\frac{d^{2\omega}q}{(q^2)^{1+v}[(q+p_1)^2-\mu^2][(q-p_2)^2-\mu^2]}$$

$$= \pi^{\omega} i^{1-2\omega} (-\mu^2)^{\omega-v-3} \Big[\Gamma(1+v)\Big]^{-1} \frac{1}{(2\pi i)^3} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} dsdtdu \Gamma(-s)\Gamma(-t)\Gamma(-u)$$
$$\times \Big(-\frac{p_1^2}{\mu^2}\Big)^s \Big(-\frac{p_2^2}{\mu^2}\Big)^t \Big(-\frac{s_1}{\mu^2}\Big)^u \frac{\Gamma(v+3-\omega+s+t+u)\Gamma(1+v+s+t)}{\Gamma(\omega+s+t+u)}$$
$$\times \frac{\Gamma(1+s+u)\Gamma(1+t+u)\Gamma(\omega-1-v+u)}{\Gamma(2+s+t+2u)}. \qquad (42)$$

Substituting the equation (42) in the equation (41), we have

$$J_3(1,1,1;\mu) = \pi^{\omega} i^{1-2\omega} (-\mu^2)^{\omega-3} \frac{1}{(2\pi i)^4} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} dvdsdtdu$$
$$\times (-\mu^2)^{-v} \Big[\Gamma(1+v)\Big]^{-1} \Gamma(-v)\Gamma(1+v)(-\mu^2)^v \qquad (43)$$
$$\times \Gamma(-s)\Gamma(-t)\Gamma(-u) \Big(-\frac{p_1^2}{\mu^2}\Big)^s \Big(-\frac{p_2^2}{\mu^2}\Big)^t \Big(-\frac{s_1}{\mu^2}\Big)^u$$
$$\times \frac{\Gamma(v+3-\omega+s+t+u)\Gamma(1+v+s+t)}{\Gamma(\omega+s+t+u)}$$
$$\times \frac{\Gamma(1+s+u)\Gamma(1+t+u)\Gamma(\omega-1-v+u)}{\Gamma(2+s+t+2u)}.$$

Utilizing Barnes formula [7]

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \Gamma(a+s)\Gamma(d+s)\Gamma(c-s)\Gamma(d-s) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}, \qquad (44)$$

we have

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dv \Gamma(v+3-\omega+s+t+u)\Gamma(v+1+s+t))\Gamma(-v)\Gamma(\omega-1+u-v)$$
$$= \frac{\Gamma(3-\omega+s+t+u)\Gamma(2+s+t+2u)\Gamma(1+s+t)\Gamma(\omega+s+t+u)}{\Gamma(3+2s+2t+2u)}, \qquad (45)$$

because $a = 3-\omega+s+t+u,\ b = 1+s+t,\ c = 0,\ d = \omega-1+u$, therefore $a+c = 3-\omega+s+t+u$, $a+d = 2+s+t+2u,\ b+c = 1+s+t,\ b+d = \omega+s+t+u$, and $a+b+c+d = 3+2s+2t+2u$. Adopting the equation (45), we can perform the v variable integration in the equation (43) first of all. Then we obtain

$$J_3(1,1,1;\mu) = \pi^{\omega} i^{1-2\omega} (-\mu^2)^{\omega-3} \frac{1}{(2\pi i)^3} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} dsdtdu$$
$$\times \Gamma(-s)\Gamma(-t)\Gamma(-u) \Big(-\frac{p_1^2}{\mu^2}\Big)^s \Big(-\frac{p_2^2}{\mu^2}\Big)^t \Big(-\frac{s_1}{\mu^2}\Big)^u$$
$$\times \frac{\Gamma(1+s+u)\Gamma(1+t+u)\Gamma(3-\omega+s+t+u)}{\Gamma(\omega+s+t+u)\Gamma(2+s+t+2u)}$$
$$\times \frac{\Gamma(2+s+t+2u)\Gamma(1++s+t)\Gamma(\omega+s+t+u)}{\Gamma(3+2s+2t+2u)} \qquad (46)$$
$$= \pi^{\omega} i^{1-2\omega} (-\mu^2)^{\omega-3} \frac{1}{(2\pi i)^3} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} dsdtdu \Gamma(-s)\Gamma(-t)\Gamma(-u)$$
$$\times \Big(-\frac{p_1^2}{\mu^2}\Big)^s \Big(-\frac{p_2^2}{\mu^2}\Big)^t \Big(-\frac{s_1}{\mu^2}\Big)^u$$
$$\times \frac{\Gamma(1+s+u)\Gamma(1+t+u)\Gamma(1+s+t)\Gamma(3-\omega+s+t+u)}{\Gamma(3+2s+2t+2u)}.$$

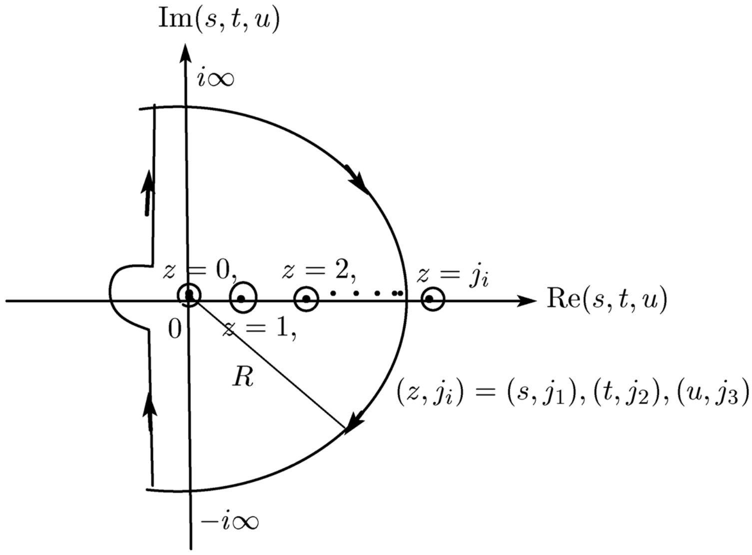

Fig.3

Next we can determine the positions of the poles in $\Gamma(-z)$ as follows:

$$\Gamma(-z)=\frac{\Gamma(j_i+1-z)}{(j_i-z)(j_i-1-z)(j_i-2-z)\ldots\ldots(-z)}, \tag{47}$$

where $(z,j_i)=(s,j_1),(t,j_2),(u,j_3)$ respectively.
We carry out the residue calculations concerning the variables s,t,u synchronously, taking the integral contours on the right half planes as shown in Fig.3. Because it is known that the contour integrals on the right semicircles of the radius R around the origin become 0 when the radius tends to ∞, the integral (46) becomes as follows:

$$J_3(1,1,1;\mu)=\pi^{\omega}i^{1-2\omega}(-\mu^2)^{\omega-3}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}\frac{\Gamma(1+j_1+j_3)\Gamma(1+j_2+j_3)\Gamma(1+j_1+j_2)}{(-1)^{j_1}j_1!(-1)^{j_2}j_2!(-1)^{j_3}j_3!}$$
$$\times\frac{\Gamma(3-\omega+j_1+j_2+j_3)}{\Gamma(3+2j_1+2j_2+2j_3)}\Big(-\frac{p_1^2}{\mu^2}\Big)^{j_1}\Big(-\frac{p_2^2}{\mu^2}\Big)^{j_2}\Big(-\frac{s_1}{\mu^2}\Big)^{j_3}. \tag{48}$$

Now as we treat the emitted real photons, we have to introduce the mass shell conditions $p_1^2=p_2^2=0$. Furthermore summing on j_1 and j_2 from 0 to ∞, $j_1=0$, $j_2=0$ terms only remain .
Therefore the equation (48) becomes

$$J_3(1,1,1;\mu)=\pi^2 i(-\mu^2)^{-1}\sum_{j_3=0}^{\infty}\frac{1}{j_3!}\Big(\frac{s_1}{\mu^2}\Big)^{j_3}\frac{\Gamma(1+j_3)\Gamma(1+j_3)\Gamma(1+j_3)}{\Gamma(3+2j_3)}$$
$$=\pi^2 i(-\mu^2)^{-1}\sum_{j_3=0}^{\infty}\frac{1}{j_3!}\Big(\frac{s_1}{\mu^2}\Big)^{j_3}\frac{(1)_{j_3}(1)_{j_3}(1)_{j_3}}{2(3)_{2j_3}}, \tag{49}$$

where we used the formula $(a)_j=a\times(a+1)\times(a+2)\times\ldots\times(a+j-1)=\Gamma(a+j)/\Gamma(a)$ and put $\omega=2$.
Using the formula [8]

$$(a)_{2n}=\Big(\frac{1}{2}a\Big)_n\Big(\frac{1}{2}a+\frac{1}{2}\Big)_n 2^{2n}, \tag{50}$$

for$(3)_{2j_3}$, we can get

$$J_3(1,1,1;\mu)=$$

$$= \pi^2 i(-\mu^2)^{-1} \sum_{j_3=0}^{\infty} \frac{1}{j_3!} \Big(\frac{s_1}{\mu^2}\Big)^{j_3} \frac{(1)_{j_3}(1)_{j_3}(1)_{j_3}}{2(\frac{3}{2})_{j_3}(2)_{j_3}} \Big(\frac{s_1}{4\mu^2}\Big)^{j_3}$$

$$= -\frac{i\pi^2}{2\mu^2} {}_3F_2\Big(1,1,1;\frac{3}{2},2;\frac{s_1}{4\mu^2}\Big), \tag{51}$$

where ${}_3F_2(a,b,c;d,e;z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j (c)_j}{j!(d)_j (e)_j} z^j$ is hypergeometric function. Adopting Clausen's formula [7]

$${}_3F_2(2a,2b,a+b;2a+2b,a+b+\frac{1}{2};z)=\Big[{}_2F_1(a,b;a+b+\frac{1}{2};z\Big]^2, \tag{52}$$

and putting $a=\frac{1}{2}, b=\frac{1}{2}$, we have

$${}_3F_2(1,1,1;\frac{3}{2},2;z)=\Big[{}_2F_1(\frac{1}{2},\frac{1}{2};\frac{3}{2};z)\Big]^2. \tag{53}$$

As the relation

$$\arcsin\sqrt{z} = \sqrt{z}\; {}_2F_1\Big(\frac{1}{2},\frac{1}{2};\frac{3}{2};z\Big) \tag{54}$$

holds [7], we can obtain the result

$$J_3(1,1,1;\mu)= -i\frac{2\pi^2}{s_1}\left(\arcsin\sqrt{\frac{s_1}{4\mu^2}}\right)^2. \tag{55}$$

This equation (55) is effective under the condition $s_1 < 4\mu^2$, that is to say, for heavy particles . In the case of section 2 or section 3 we can't use the equation (55) because of $s_1 > 4\mu^2$. We'll need to figure out a good way to handle such a situation by taking the other integral contours or doing analytic continuation to the region $s_1 < 4\mu^2$ when we integrate the equation (46) .

5 Concluding Remarks

In this paper we tried to calculate Feynman integral in two ways to establish the integral domain. While calculating it and using our new parameter transformation, we found the new regularization of Feynman integral concerning the divergences from the integral lower limit $x_i = 0$. Furthermore the result of the estimation of the ratio $\Gamma(K_S^0 \to 2\gamma)/\Gamma(K_S^0 \to \pi^+\pi^-)$ is consistent with the experimental data using the integral domain $0 \le x_3 \le 1$, but it's not consistent with experimental data using the integral domain $0 \le x_3 \le 1 - x_2$. However our consideration is rough, so more rigorous considerations will be needed. Furthermore we tried to calculate the same Feynman integral by utilizing Davydychev method . The calculation by this method is independent of the integral domain immediately besides the integral contours. Because the result of the calculation is expressed by using hypergeometric function, we can do analytic continuation to some domains easily, so that we'll be able to examine the domain problem between the integral domains in section 2 and in section 3 by using the result of the calculation by Davydychev method.

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