The extension to the left half semicircle contour and the analytic continuation technique in Davydychev method calculation of our Feynman integrals

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Abstract

In the previous paper we showed that Davydychev method is useful when we calculate our Feynman integrals. The results of the calculation are valid in the domains$s_1/4\mu^2 < 1$. In this paper we try to do analytic continuation to the domain $4\mu^2/s_1 < 1$. Furthermore we consider the way to calculate our Feynman integrals by taking the integral contour of semicircle on the left half plane. This calculation is complex a bit because there are many singularities.

Key Words : Feynman integrals, quantum chromodynamics, hyper-geometric function

1 Introduction

In the previous papers we have showed the new parameter transformation and the integral techniques of Feynman integrals.[3][4] This new parameter transformation was what we solved Schwinger parametrization equation in reverse and generalized it.[1][2] Furthermore we showed that the method to utilize Hyper geometric function is useful on the occasion of calculation of Feynman propagator. The advantages to make use of Hyper geometric function are as follows, (i) we can describe the results of calculation by means of the convergent series, (ii) we can get a few kind of integral representations of Hyper geometric function easily, (iii) we can do the analytic continuation of Hyper geometric function from one domain to another domain of variable easily.[5][6][7] As a result we can study the mathematical and physical features of the solutions in the several regions. (iv) Hyper geometric functions include the important functions which appear in physics, for example, trigonometric function, spherical function, and so on, as the sub functions. The difficult points are (i) Sometimes the calculations are so complex rather than ones by using the usual functions. (ii) There are a lot of undeveloped portions, especially, in Kampè de Fèriet function, Lauricella function, and so on. But in these days from the calculations of Feynman propagator the recurrence formulas and the differential equations which Hyper geometric function obeys are being found. Last year in the previous paper we calculated the Feynman propagator in $K_S^0 \to 2\gamma$ process by using Davydychev method.[4] But we obtained the result that became effective on the condition $s_1 < 4\mu^2$, of the so-called heavy particles case. Namely it's not effective in $K_S^0 \to 2\gamma$ process. In this paper we'd like to obtain the consequences and the way of the calculation to lead the effective result for light particles. In section.**2** we review the Davydychev method in the previous paper.[4][8][9] In section.**3** we calculate Feynman propagator by using contour integral on the left half complex plane. In section.**4** we calculate the same in-

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tegral by using analytic continuation and double limit procedure of Hyper geometric function ${}_2F_1(a,b;c;z)$. In section.**5** we discuss the result and propose the further progress of the calculation in this paper as concluding remarks.

2 The calculation of Feynman integral using Davydychev method

At first we'd like to review the calculation method of Feynman propagators shown by Davydychdev and others, and that I wrote in my paper last year.[4] When we calculate the decay width in the decay process $K^0_S \to 2\gamma$, we have to evaluate the following integral finally,

$$J_3(1,1,1;\mu) = \int \frac{d^{2\omega}q}{[q^2-\mu^2+i\epsilon][(q+p_1)^2-\mu^2+i\epsilon][(q-p_2)^2-\mu^2+i\epsilon]}, \tag{1}$$

where p_1 and p_2 are four momentum of emited final real photons respectively, and μ^2 is pion mass squared. In such a process the mass shell condition $p_1^2 = p_2^2 = 0$ is satisfied. Getting rid of infinitesimal quantity $i\epsilon$, we perform the calculation in pseudo Euclidean mometum space. Furthermore the dimensional parameter is $\omega = 2-\epsilon$. Using Hyper geometric function ${}_1F_0\Big(1,\frac{\mu^2}{q^2}\Big)$, we can express Eq. (1) as

$$J_3(1,1,1;\mu) = \int \frac{1}{q^2}{}_1F_0\left(1;\frac{\mu^2}{q^2}\right)\frac{d^{2\omega}q}{[(q+p_1)^2-\mu^2][(q-p_2)^2-\mu^2]}. \tag{2}$$

On exploiting Mellin Barnes integral representation of hyper geometric function

$${}_1F_0\left(1;\frac{\mu^2}{q^2}\right) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(-v)\Gamma(1+v)(-\mu^2)^v\,dv, \tag{3}$$

Eq. (1) becomes

$$J_3(1,1,1;\mu) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dv\Gamma(-v)\Gamma(1+v)\left(-\frac{\mu^2}{q^2}\right)^v J_2(1+v,1,1;\mu), \tag{4}$$

where $J_2(1+v,1;1,\mu)$ is

$$\begin{aligned}
J_2(1+v,1,1+\mu) &= \int \frac{d^{2\omega}q}{(q^2)[(q+p_1)^2-\mu^2][(q-p_1)^2-\mu^2]} \\
&= \pi^{\omega} i^{1-2\omega}\Big[\Gamma(1+v)\Big]^{-1}\frac{1}{(2\pi i)^3}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} ds\,dt\,du\Gamma(-s)\Gamma(-t)\Gamma(-u) \\
&\times\left(-\frac{p_1^2}{\mu^2}\right)^s\left(-\frac{p_2^2}{\mu^2}\right)^t\left(-\frac{s_1}{\mu^2}\right)^u\frac{\Gamma(v+3-\omega+s+t+u)\Gamma(1+v+s+t)}{\Gamma(\omega+s+t+u)} \\
&\times\frac{\Gamma(1+s+u)\Gamma(1+t+u)\Gamma(\omega-1-v+u)}{\Gamma(2+s+t+2u)}.
\end{aligned} \tag{5}$$

Substituting Eq. (5) in Eq. (4) yields

$$\begin{aligned}
J_3(1,1,1;\mu) &= \pi^{\omega} i^{1-2\omega}(-\mu^2)^{\omega-3}\frac{1}{(2\pi i)^4}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} dv\,ds\,dt\,du \\
&\times(-\mu^2)^{-v}\Big[\Gamma(1+v)\Big]^{-1}\Gamma(-v)\Gamma(1+v)(-\mu^2)^v \\
&\times\Gamma(-s)\Gamma(-t)\Gamma(-u)\left(-\frac{p_1^2}{\mu^2}\right)^s\left(-\frac{p_2^2}{\mu^2}\right)^t\left(-\frac{s_1}{\mu^2}\right)^u \\
&\times\frac{\Gamma(v+3-\omega+s+t+u)\Gamma(1+v+s+t)}{\Gamma(\omega+s+t+u)} \\
&\times\frac{\Gamma(1+s+u)\Gamma(1+t+u)\Gamma(\omega-1-v+u)}{\Gamma(2+s+t+2u)}.
\end{aligned} \tag{6}$$

In this stage, utilizing Barnes formula concerning Γ function[10]

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}, \tag{7}$$

and further performing the integration concerning variable v, we can obtain a beautiful formula symmetrical to variables s, t, u,

$$\begin{aligned} J_3(1,1,1;\mu) &= \pi^{\omega} i^{1-2\omega}(-\mu^2)^{\omega-3}\frac{1}{(2\pi i)^3}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} ds\, dt\, du\Gamma(-s)\Gamma(-t)\Gamma(-u) \\ &\times\left(-\frac{p_1^2}{\mu^2}\right)^{s}\left(-\frac{p_2^2}{\mu^2}\right)^{t}\left(-\frac{s_1}{\mu^2}\right)^{u} \\ &\times\frac{\Gamma(1+s+u)\Gamma(1+t+u)\Gamma(1+s+t)\Gamma(3-\omega+s+t+u)}{\Gamma(3+2s+2t+2u)}, \end{aligned} \tag{8}$$

where $s_1 = (p_1 - p_2)^2$ is kaon mass squared of input particle.
Now we carry out the residue calculations concerning variables s, t, u synchronously, taking the integral contours on the right half complex planes. In these planes $\Gamma(-s), \Gamma(-t), \Gamma(-u)$ only have single poles on the right half complex planes.
The result of the residue calculation becomes

$$\begin{aligned} J_3(1,1,1;\mu) &= \pi^{\omega} i^{1-2\omega}(-\mu^2)^{\omega-3}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}\frac{\Gamma(1+j_1+j_3)\Gamma(1+j_2+j_3)\Gamma(1+j_1+j_2)}{(-1)^{j_1}j_1!(-1)^{j_2}j_2!(-1)^{j_3}j_3!} \\ &\times\frac{\Gamma(3-\omega+j_1+j_2+j_3)}{\Gamma(3+2j_1+2j_2+2j_3)}\left(-\frac{p_1^2}{\mu^2}\right)^{j_1}\left(-\frac{p_2^2}{\mu^2}\right)^{j_2}\left(-\frac{s_1}{\mu^2}\right)^{j_3}. \end{aligned} \tag{9}$$

Because we treat emitted real photons, we have $p_1^2 = p_2^2 = 0$ from the mass shell condition. So only the $j_1 = 0, j_2 = 0$ terms survive in Eq.(9). Now we can take $\omega = 2$. This means the analytic continuation from 2ω dimension to four dimension. Therefore we have

$$J_3(1,1,1;\mu) = \pi^2 i(-\mu^2)^{-1}\sum_{j_3=0}^{\infty}\frac{1}{j_3!}\left(\frac{s_1}{\mu^2}\right)^{j_3}\frac{\Gamma(1+j_3)\Gamma(1+j_3)\Gamma(1+j_3)}{\Gamma(3+2j_3)}. \tag{10}$$

And on using Hyper geometric function, we have

$$\begin{aligned} J_3(1,1,1;\mu) &= \pi^2 i(-\mu^2)^{-1}\sum_{j_3=0}^{\infty}\frac{1}{j_3!}\left(\frac{s_1}{\mu^2}\right)^{j_3}\frac{(1)_{j_3}(1)_{j_3}(1)_{j_3}}{2\left(\frac{3}{2}\right)_{j_3}(2)_{j_3}}\left(\frac{s_1}{4\mu^2}\right)^{j_3} \\ &= -\frac{i\pi^2}{2\mu^2}{}_3F_2\left(1,1,1;\frac{3}{2},2;\frac{s_1}{4\mu^2}\right). \end{aligned} \tag{11}$$

Adopting Claussen's formula[10]

$${}_3F_2\left(1,1,1;\frac{3}{2},2;\frac{s_1}{4\mu^2}\right) = \left[{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{s_1}{4\mu^2}\right)\right]^2, \tag{12}$$

and

$$\arcsin\sqrt{\frac{s_1}{4\mu^2}} = \sqrt{\frac{s_1}{4\mu^2}}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{s_1}{4\mu^2}\right), \tag{13}$$

we can obtain the following final result

$$J_3(1,1,1;\mu) = -i\frac{2\pi^2}{s_1}\left(\arcsin\sqrt{\frac{s_1}{4\mu^2}}\right)^2. \tag{14}$$

This equation is effective on the condition $s_1 < 4\mu^2$. In this case because of $s_1 = 0.248$GeV(Kaon mass squared) and $\mu^2 = 0.0195$GeV(Pion mass squared) the condition is not satisfied. Therefore we need to find out a good way to handle such a situation. In the following sections we'd like to show the good ways by taking the other integral contour or doing the analytic continuation to the region $s_1 > 4\mu^2$ when we evaluate Eq. (8).

3 The calculation by using contour integral on the left half complex plane

As we reviewed in the previous section , the result of the calculation was effective on the condition $s_1 < 4\mu^2$. But at the calculation in phenomenon $K_0^S \to 2\gamma$ we have to obtain the result of the calculation which is effective in the region $s_1 > 4\mu^2$. To get such a effective result, we must extend the integral domain to the left half complex plane in this case of the calculation of ${}_2F_1(1/2, 1/2; 3/2; s_1/4\mu^2)$. By doing that, we can gain the formula effective in the region $s_1 > 4\mu^2$. First of all Mellin Barnes integral representation of hyper geometric function ${}_2F_1(a, b; c; z)$ is expressed as follows,

$$
{}_2F_1(a,b;c;z) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s \, ds. \tag{15}
$$

Applying this formula to hyper geometric function ${}_2F_1(1/2, 1/2; 3/2; s_1/4\mu^2)$, we have the following formula

$$
\begin{aligned}
&{}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{s_1}{4\mu^2}\Big) \\
&= \frac{\Gamma(3/2)}{\Gamma(1/2)\Gamma(1/2)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma\Big(\frac{1}{2}+t\Big)\Gamma\Big(\frac{1}{2}+t\Big)\Gamma(-t)}{\Gamma\Big(\frac{3}{2}+t\Big)} \Big(-\frac{s_1}{4\mu^2}\Big)^t dt \\
&= \frac{1}{2\sqrt{\pi}} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma\Big(\frac{1}{2}+t\Big)\Gamma\Big(\frac{1}{2}+t\Big)\Gamma(-t)}{\Gamma\Big(\frac{3}{2}+t\Big)} \Big(-\frac{s_1}{4\mu^2}\Big)^t dt.
\end{aligned} \tag{16}
$$

Of course, doing contour integral on the right half complex plane returns Eq. (16) to the original hyper geometric function ${}_2F_1(1/2, 1/2; 3/2; s_1; 4\mu^2)$ again because of the contribution of the poles from $\Gamma(-t)$ only. This situation is the same as the consideration of section 2 entirely.
Now we perform contour integral on the left half complex plane as drawing the contour in Fig.1, and we show that we can derive the formula effective in the region $s_1 > 4\mu^2$ by doing such a contour integration. The positions of poles in the integrand are shown in Fig.1. It is understood from the following formula

$$
\Gamma\Big(\frac{1}{2}+t\Big) = \frac{\Gamma\Big(\frac{1}{2}+t+n+1\Big)}{\Big(\frac{1}{2}+t+n\Big)\Big(\frac{1}{2}+t+n-1\Big)\cdots\cdots\Big(\frac{1}{2}+t\Big)}, \tag{17}
$$

that the integrand has double poles , substituting Eq. (17) to Eq. (16), that is,

$$
\begin{aligned}
&{}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{s_1}{4\mu^2}\Big) \\
&= \frac{1}{2\sqrt{\pi}} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Big(\Gamma\Big(\frac{1}{2}+t+n+1\Big)\Big)^2 \Gamma(-t)}{\Gamma\Big(\frac{3}{2}+t\Big)\Big(\frac{1}{2}+t+n\Big)^2\Big(\frac{1}{2}+t+n-1\Big)^2\cdots\cdots\Big(\frac{1}{2}+t\Big)^2} \Big(-\frac{s_1}{4\mu^2}\Big)^t dt.
\end{aligned} \tag{18}
$$

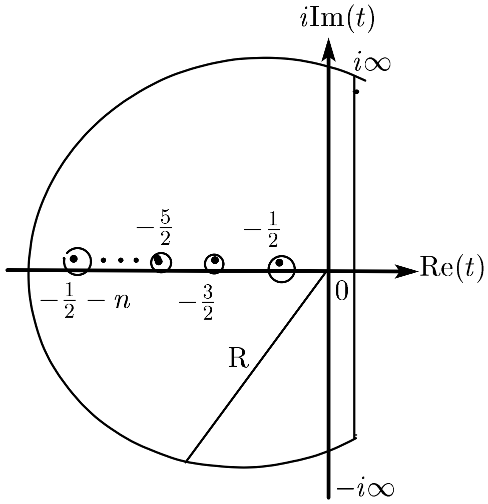

Fig.1

The residue of integral concerning the double poles is calculated as follows,

$$\mathrm{residue}\Big[t=-n-\frac{1}{2};\mathrm{Integrand}\Big]=\lim_{t\to -n-\frac{1}{2}}\Big[\frac{d}{dt}\Big(\big(t+n+1/2\big)^2\mathrm{Integrand(t,n)}\Big)\Big]$$

$$=-\frac{d}{dn}\Bigg[\frac{\Gamma(n+1/2)}{\Gamma(1-n)\Big((-1)(-2)\cdot\cdots\cdot(-n)\Big)^2}\Big(-\frac{s_1}{4\mu^2}\Big)^{-n-\frac{1}{2}}\Bigg] \tag{19}$$

$$=-\frac{d}{dn}\Bigg[\frac{\Gamma(n+1/2)}{\Gamma(1-n)((-1)^n)^2(n!)^2}\Big(-\frac{s_1}{4\mu^2}\Big)^{-n-\frac{1}{2}}\Bigg]$$

$$=i\sqrt{\frac{4\mu^2}{s_1}}\Bigg[\frac{(-1)^n\Gamma(n+\frac{1}{2})}{\Gamma(1-n)(\Gamma(n+1))^2}\Bigg]\Bigg(\psi(1-n)+2\psi(n+1)-\psi(n+1/2)-\log\Big(\frac{4\mu^2}{s_1}\Big)\Bigg)\Big(\frac{4\mu^2}{s_1}\Big)^n.$$

Making use of this residue formula, we can find ${}_2F_1(1/2,1/2;3/2;4\mu^2/s_1)$ as follows,

$${}_2F_1\Big(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{4\mu^2}{s_1}\Big)=\frac{i}{2\sqrt{\pi}}\sqrt{\frac{4\mu^2}{s_1}}\sum_{n=0}^{\infty}\Bigg[\frac{(-1)^n\Gamma(n+1/2)}{\Gamma(1-n)(\Gamma(n+1))^2}\Bigg]$$
$$\times\Bigg(\psi(1-n)+2\psi(n+1)-\psi(n+1/2)-\log\Big(\frac{4\mu^2}{s_1}\Big)\Bigg)\Big(\frac{4\mu^2}{s_1}\Big)^n=F_A\Big(\frac{4\mu^2}{s_1}\Big), \tag{20}$$

where $\psi(z)=\frac{d}{dz}\log\Gamma(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ is called Psi function.

Finally we can gain the following final result

$$J_3(1,1,1;\mu)=-\frac{i\pi^2}{2\mu^2}\Bigg[F_A\Bigg(\frac{4\mu^2}{s_1}\Bigg)\Bigg]^2. \tag{21}$$

We understand that Eq. (21) is effective in the region $s_1>4\mu^2$. Actually we could obtain the solution effective in the region $s_1>4\mu^2$ by carrying out contour integral in Mellin Barnes integral representation of ${}_2F_1(1/2,1/2;3/2;s_1/4\mu^2)$ on the left half complex plane. Because $\Gamma(1-n)$ tends to ∞ except $n=0$ and so the first term only ($n=0$ term) survives in Eq. (21), the equation (21) becomes as follows, $J_3(1,1,1;\mu)=\frac{i\pi^2}{8\mu^2}\Big(\frac{4\mu^2}{s_1}\Big)\Big(2\gamma-2\log 2+\log(\frac{4\mu^2}{s_1})\Big)^2$, where γ is Euler's constant and we used $\psi(1)=-\gamma$, $\psi(1/2)=-\log 4-\gamma$.[10]

We try to integrate Eq. (8) directly by using contour integral on the left half plane. At this time

considering the mass shell condition of final photons $p_1^2 = p_2^2 = 0$, it is better for integrating on the right half plane with variables s, and, t, and after that, integrating on the left half plane with variable u.
From Eq. (8) we have again, putting $\omega = 2$, the following formula

$$J_3(1,1,1;\mu) = \pi^2(-\mu^2)^{-1}\frac{i}{(2\pi i)^3}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} ds\,dt\,du \tag{22}$$
$$\times\Gamma(-s)\Gamma(-t)\Gamma(-u)\frac{\Gamma(1+s+u)\Gamma(1+t+u)\Gamma(1+s+t)\Gamma(1+s+t+u)}{\Gamma(3+2s+2t+2u)}$$
$$\times\Big(-\frac{p_1^2}{\mu^2}\Big)^s\Big(-\frac{p_2^2}{\mu^2}\Big)^t\Big(-\frac{s_1}{\mu^2}\Big)^u.$$

Integrating with respect to variables s, t, we have

$$J_3(1,1,1;\mu) = \frac{-\pi^2}{\mu^2}\frac{i}{2\pi i}\int_{-i\infty}^{i\infty} du \sum_{j_1,j_2=0}^{\infty}\Gamma(-u) \tag{23}$$
$$\times\frac{\Gamma(1+j_1+u)\Gamma(1+j_2+u)\Gamma(1+j_1+j_2)\Gamma(1+j_1+j_2+u)}{(-1)^{j_1}(-1)^{j_2}j_1!j_2!\Gamma(3+2j_1+2j_2+2u)}$$
$$\times\Big(-\frac{p_1^2}{\mu^2}\Big)^{j_1}\Big(-\frac{p_2^2}{\mu^2}\Big)^{j_2}\Big(-\frac{s_1}{\mu^2}\Big)^u.$$

Because we have the relation

$$\Gamma(1+u) = \frac{\Gamma(1+u+j+1)}{(1+u+j)(1+u+j-1)\cdot\cdots\cdot(1+u)}, \tag{24}$$

Eq. (23) has the triple poles as $j_1 = j_2 = 0$.
Carrying out the residue calculation, we obtain as follows,

$$J_3(1,1,1,\mu) = -\frac{\pi^2}{\mu^2}\frac{1}{2!}\left(\frac{d^2}{du^2}\Big[(u+j_3+1)^3\text{Integrand}\Big]\right)_{u=-j_3-1}. \tag{25}$$

We get the following final result

$$J_3(1,1,1,\mu) = -\frac{\pi^2}{\mu^2}\frac{1}{2!}\left[\frac{d^2}{dj_3^2}\frac{(-1)^{j_3}}{(j_3!)^3\Gamma(1-2j_3)}\Big(-\frac{s_1}{\mu^2}\Big)^{-j_3-1}\right]. \tag{26}$$

It is tedious a bit to evaluate Eq. (26) in detail.

4 The analytic continuation method

Hyper geometric function ${}_2F_1(a,b;c;z)$ defined by

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}z^n \tag{27}$$

converges on the condition of $|z| < 1$. Now we consider the analytic continuation of ${}_2F_1(a,b;c;z)$ to the region $|\frac{1}{z}| < 1$. By taking such a situation we can make the formula which holds in the region $s_1 > 4\mu^2$.
The analytic continuation formula of ${}_2F_1(a,b;c;z)$ is expressed as follows,[10]

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1(a,a-c+1;a-b+1:1/z)$$
$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1(b,b-c+1;b-a+1:1/z). \tag{28}$$

Applying this formula to ${}_2F_1(1/2,1/2;3/2;z)$,because of $\Gamma(a-b)=\Gamma(b-a)=\Gamma(0)=\infty$ we can't obtain the meaningful formula by analytic continuation of ${}_2F_1(1/2,1/2;3/2;z)$. To get rid of this difficulty we introduce an idea of double limit procedure as follows,

$$ {}_2F_1\Big(\frac{1}{2},\frac{1}{2};\frac{3}{2};z\Big)=\lim_{\epsilon\to 0}{}_2F_1\Big(\frac{1}{2}+\epsilon,\frac{1}{2}-\epsilon;\frac{3}{2};z\Big). \tag{29} $$

By using this procedure, ${}_2F_1(1/2,1/2;3/2;z)$ becomes as follows,

$$ {}_2F_1\Big(\frac{1}{2},\frac{1}{2};\frac{3}{2};z\Big)=\lim_{\epsilon\to 0}\Big[{}_2F_1\Big(\frac{1}{2}+\epsilon,\frac{1}{2}-\epsilon;\frac{3}{2};z\Big)\Big]=\lim_{\epsilon\to 0}\Big[\frac{\Gamma(\frac{3}{2})\Gamma(-2\varepsilon)}{\Gamma(\frac{1}{2}-\varepsilon)\Gamma(1-\epsilon)}\Big(-\frac{1}{z}\Big)^{\frac{1}{2}+\varepsilon} \tag{30} $$

$$ \times{}_2F_1(\frac{1}{2}+\epsilon,\epsilon;1+2\epsilon:1/z)+\frac{\Gamma(\frac{3}{2})\Gamma(2\epsilon)}{\Gamma(\frac{1}{2}+\epsilon)\Gamma(1+\epsilon)}\Big(-\frac{1}{z}\Big)^{\frac{1}{2}-\epsilon}{}_2F_1(\frac{1}{2}-\epsilon,-\epsilon;1-2\epsilon:1/z)\Big] $$

Furthermore we utilize the following formulas[11]

$$ \Gamma(-2\epsilon)=\frac{1}{-2\epsilon}-\gamma+(\gamma^2+\pi^2/6)(-\epsilon)+O(\epsilon^2)\ \text{ and }\ \Gamma(2\epsilon)=\frac{1}{2\epsilon}-\gamma+(\gamma^2+\pi^2/6)\epsilon+O(\epsilon^2) \tag{31} $$

$$ \lim_{\epsilon\to 0}{}_2F_1\Big(\frac{1}{2}+\epsilon,\epsilon;1+2\epsilon;\frac{1}{z}\Big)={}_2F_1\Big(\frac{1}{2},0;1:\frac{1}{z}\Big)=1, \tag{32} $$

$$ \lim_{\epsilon\to 0}{}_2F_1\Big(\frac{1}{2}-\epsilon,-\epsilon;1-2\epsilon;\frac{1}{z}\Big)={}_2F_1\Big(\frac{1}{2},0;1:\frac{1}{z}\Big)=1, \tag{33} $$

$$ \Big(-\frac{1}{z}\Big)^{\frac{1}{2}+\epsilon}=i\Big(\frac{1}{z}\Big)^{\frac{1}{2}}\Big(-\frac{1}{z}\Big)^{\epsilon}=\frac{i(-1)^\epsilon}{\sqrt{z}}\exp(\epsilon\log\frac{1}{z})=\frac{i(-1)^\epsilon}{\sqrt{z}}\Big[1+(\log\frac{1}{z})\epsilon+O(\epsilon^2)\Big], \tag{34} $$

and similarly

$$ \Big(-\frac{1}{z}\Big)^{\frac{1}{2}-\epsilon}=\frac{i(-1)^\epsilon}{\sqrt{z}}\Big[1-(\log\frac{1}{z})\epsilon+O(\epsilon^2)\Big]. \tag{35} $$

Substituting Eqs. (31)(32)(33)(34)(35) to Eq. (30) yields a meaningful formula analytically continued from the region $|z|<1$ to the region $|z|>1$.
That is,

$$ \begin{aligned} {}_2F_1\Big(\frac{1}{2},\frac{1}{2};\frac{3}{2};z\Big)&=\lim_{\epsilon\to 0}{}_2F_1\Big(\frac{1}{2}+\epsilon,\frac{1}{2}-\epsilon;\frac{3}{2};z\Big)\\ &=\lim_{\epsilon\to 0}\Big[\frac{\Gamma(3/2)\Gamma(-2\epsilon)}{\Gamma(1/2-\epsilon)\Gamma(1-\epsilon)}\Big(-\frac{1}{z}\Big)^{\frac{1}{2}+\epsilon}{}_2F_1\Big(\frac{1}{2}+\epsilon,\epsilon;1+2\epsilon;\frac{1}{z}\Big)\\ &+\frac{\Gamma(3/2)\Gamma(2\epsilon)}{\Gamma(1/2+\epsilon)\Gamma(1+\epsilon)}\Big(-\frac{1}{z}\Big)^{\frac{1}{2}-\epsilon}{}_2F_1\Big(\frac{1}{2}-\epsilon,-\epsilon;1-2\epsilon;\frac{1}{z}\Big)\Big]\Big]\\ &=\frac{\Gamma(3/2)}{\Gamma(1/2)\Gamma(1)}{}_2F_1(\frac{1}{2},0;1;\frac{1}{z})\lim_{\epsilon\to 0}\Big[\Gamma(-2\epsilon)\Big(-\frac{1}{z}\Big)^{\frac{1}{2}+\epsilon}+\Gamma(2\epsilon)\Big(-\frac{1}{z}\Big)^{\frac{1}{2}-\epsilon}\Big] \end{aligned} \tag{36} $$

Furthermore we can show that the divergent terms cancel each other by exploiting the double limit procedure as follows,

$$ \begin{aligned} {}_2F_1(\frac{1}{2},\frac{1}{2};\frac{3}{2};z)&=\frac{1}{2}\lim_{\epsilon\to 0}\Big[\Big(\frac{1}{-2\epsilon}-\gamma+(\gamma^2+\pi^2/6)(-\epsilon)+O(\epsilon^2)\Big)\frac{i(-1)^\epsilon}{\sqrt{z}}\Big(1+(\log(1/z))\epsilon+O(\epsilon^2)\Big)\\ &+\Big(\frac{1}{2\epsilon}-\gamma+(\gamma^2+\pi^2/6)(\epsilon)+O(\epsilon^2)\Big)\frac{i(-1)^\epsilon}{\sqrt{z}}\Big(1-(\log(1/z))\epsilon+O(\epsilon^2)\Big)\Big]\\ &=\frac{i}{2\sqrt{z}}\lim_{\epsilon\to 0}\Big[-\frac{1}{2\epsilon}-\frac{1}{2}\log(\frac{1}{z})-\gamma+O(\epsilon)+\frac{1}{2\epsilon}-\frac{1}{2}\log(\frac{1}{z})-\gamma+O(\epsilon)\Big]\\ &=-\frac{i}{2\sqrt{z}}\Big[2\gamma+\log\Big(\frac{1}{z}\Big)\Big]. \end{aligned} \tag{37} $$

Because

$$ {}_3F_2(1,1,1;\frac{1}{2},\frac{3}{2})=\Big[{}_2F_1(1/2,1/2;3/2)\Big]^2=-\frac{1}{4z}\Big(2\gamma+\log\frac{1}{z}\Big)^2, \tag{38} $$

we have

$$ J_3(1,1,1;\mu)=-\frac{i\pi^2}{2\mu^2}\Big[-\frac{1}{4z}\Big(2\gamma+\log\frac{1}{z}\Big)^2\Big]=\frac{i\pi^2}{8\mu^2}\frac{1}{z}\Big(2\gamma+\log\frac{1}{z}\Big)^2. \tag{39} $$

Putting $z=s_1/4\mu^2$, we obtain the formula effective in the region $s_1>4\mu$ as follows,

$$ J_3(1,1,1,\mu)=\frac{i\pi^2}{8\mu^2}\Big(\frac{4\mu^2}{s_1}\Big)\Big(2\gamma+\log\Big(\frac{4\mu^2}{s_1}\Big)\Big)^2. \tag{40} $$

Comparing Eq. (40) to Eq. (21), we know that Eq. (40) is consistent with the Eq. (21) apart from an additional constant $-2\log 2$.

Now let us prove the following formula that we used when deriving Eq. (39) in the two ways,

$$ {}_2F_1\Big(\frac{1}{2},0;1;z\Big)={}_2F_1\Big(0,\frac{1}{2};1;z\Big)=1. \tag{41} $$

Euler's integral representation of Hyper geometric function ${}_2F_1(a,b;c;z)$ is expressed as follows,

$$ {}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt. \tag{42} $$

(i) The way that we integrate Eq. (41) immediately ;

$$ \begin{aligned} {}_2F_1\Big(0,\frac{1}{2};1;z\Big)&=\frac{\Gamma(1)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt\\ &=\frac{1}{\sqrt{\pi}\sqrt{\pi}}\int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt=\frac{1}{\pi}\int_0^1\frac{dt}{\sqrt{t-t^2}}=\frac{1}{\pi}\Big[-\arcsin(-2z+1)\Big]_0^1\\ &=\frac{1}{\pi}\Big(-\arcsin(-1)+\arcsin(1)\Big)=\frac{1}{\pi}\Big(-(-\frac{\pi}{2})+\frac{\pi}{2}\Big)=\frac{1}{\pi}\times\pi=1 \end{aligned} \tag{43} $$

(ii) the way that we use the definition of beta function ;

$$ \begin{aligned} {}_2F_1(0,b;c;z)&=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}dt\\ &=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}B(b,c-b)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\times\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}=1 \end{aligned} \tag{44} $$

Therefore we can show that ${}_2F_1(1/2,0;1;z)=1$ holds.

5 Concluding Remarks

In this paper we reviewed the calculation method of Feynman propagator which appears when we calculate the decay width of decay process $K_S^0\to 2\gamma$, and discovered by Davydychev and others. The result of the calculation was effective on the condition $s_1<4\mu^2$. But in this case we must have the formula which holds on the condition $s_1>4\mu^2$. In this paper we proposed two ways of the calculation by which we can obtain the result effective on the condition $s_1>4\mu^2$. The first way was the method of the calculation that we employ contour integral on the left half complex plane, different from the contour integral in the previous paper. At that time it is known that the double poles appear in the integrand. By using this method of calculation we could gain the formula effective in the region $s_1>4\mu^2$. But it was that we integrated the formula of Mellin Barnes representation of ${}_2F_1(a,b;c;z)$ on the contour of the left half complex plane. I think that we have to perform the triple integral of Eq. (8) on the contour of the left half complex plane directly. In this situation three cases are considered when we do the contour integrals of Eq. (8), that is, (i) on the right half plane with respect to variables s, t and on the left half plane with variable u. (ii) on the right half plane with one of variables s, t, u and on the left half plane with remaining two variables. (iii) on the left half plane with respect to variables s, t, u all. In the cases of (ii) and (iii) the calculations may cause a few problems as convergence, integrability, and so on. The second way of the calculation was the method that we use analytic continuation of Hyper geometric function. In this case we had to carry out the analytic continuation from the region $|z| < 1$ to the region $|\frac{1}{z}| < 1$. We could not get the meaningful formula because $\Gamma(0) = \infty$ appeared in the formula. Therefore we proposed double limit procedure technique. We could obtain the meaningful result by using this technique. But this result is not consistent with the one of section 3 by the contour integral on the left half plane completely. We may be able to understand why the difference between both results by two methods yields by considering other contour of the integration, for example, as Pochhammer's contour.

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