An Algorithm for The Feedback Vertex Set Problem in a Certain Class of Circular-arc Graphs

Hirotoshi HONMA[1] Yoko NAKAJIMA[1]

Abstract: In an undirected graph, the feedback vertex set (FVS) problem is to find the set of vertices of minimum cardinality whose removal renders the graph acyclic. The FVS problem has applications in several areas such as combinatorial circuit design, synchronous systems, computer systems, and very-large-scale integration (VLSI) circuits. The FVS problem is known to be NP-hard for simple graphs, but interesting polynomial-time solutions have been found for special classes of graphs. The intersection graph of a collection of arcs on a circle is called a circular-arc graph. A normal Helly circular-arc graph is a proper subclass of the set of circular-arc graphs. In this paper, we present an algorithm that takes $O(n+m)$ time to solve the FVS problem in a normal Helly circular-arc graph with n vertices and m edges

Key words: Design and analysis of algorithms, graph theory, feedback vertex set, normal Helly circular-arc graphs, intersection graphs;

]

1 Introduction

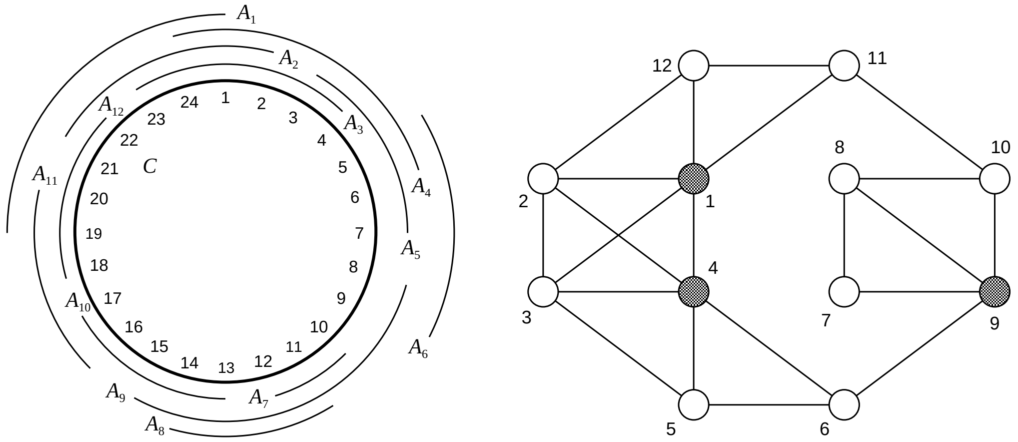
Let $\mathscr{F}$ be a family of nonempty sets. A simple graph G is the *intersection graph* of $\mathscr{F}$ if there exists a one-to-one correspondence between the vertices of G and the sets in $\mathscr{F}$, such that two vertices in G are adjacent if and only if their corresponding sets have a nonempty intersection. If $\mathscr{F}$ is a family of intervals on the real line, G is called an *interval graph* [1]. Furthermore, a graph G is called a *circular-arc graph* if it is the intersection graph of a collection of arcs on a circle [1]. Circular-arc graphs properly contain a class of interval graphs as a subclass. Circular-arc graphs have applications in areas such as genetics [2], traffic control [1], multidimensional scaling [3], compiler design [4], ring network modeling [5]. In recent years, circular-arc graphs have been investigated extensively from both theoretical and algorithmic perspectives [6, 7, 8, 9].

Let $G = (V, E)$ be a simple graph, where V is the set of vertices and E is the set of edges of G, with $|V| = n$ and $|E| = m$. Suppose that V' is a nonempty subset of V. The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both vertices in V' is called the *induced subgraph* on V' and is denoted by $G[V']$ [10]. A cycle with no repeated vertices is a simple cycle. In this paper, the term "cycle" denotes "simple cycle." A *feedback vertex set* (FVS) consists of a subset $F \subseteq V$ such that each cycle in G contains at least one vertex in F. In other words, a subset $F \subseteq V$ is an FVS of G if the subgraph induced by $G[V - F]$ is acyclic. The FVS problem is to find an FVS of minimum cardinality (MFVS) in G. The FVS problem has applications in several areas such as deadlock prevention in operating systems [11], combinatorial circuit design [12], VLSI circuits [13], and information security [14].

The FVS problem is known to be NP-hard for general graphs [15] and bipartite graphs [16]. In general, it is known that more efficient algorithms can be developed by restricting classes of graphs. For instance, interesting polynomial-time solutions for the FVS problem have been found for special classes of graphs, such as interval graphs [17, 18], permutation graphs [19], butterfly networks [20], hypercubes [21], star graphs [22], diamond graphs [23], and rotator graphs [24]. Saha and Pal presented an algorithm that takes $O(n+m)$ time for the FVS problem in interval graphs using maximal clique decomposition [18]. The algorithm obtains an MFVS in an interval graph by breaking all cycles for each maximal clique. Circular-arc graphs are a natural generalization of interval graphs. However, the algorithm presented by Saha and Pal [18] cannot be directly applied to circular-arc graphs because the number of maximal cliques in interval graphs is at most the number of vertices, whereas circular-arc graphs may have an exponential number of maximal cliques [25].

Normal Helly circular-arc models (NHCM) are precisely those without three or less arcs covering the whole circle [26, 27]. Caimi et al. showed that the number of maximal cliques is at most n for any normal Helly circular-arc graph with n vertices and m edges, and that all maximal cliques can be easily found in $O(n + m)$ time [28]. In this study, we pro-

[1]Department of Information Engineering, National Institute of Technology, Kushiro College



(a) A normal Helly circular-arc model CM_1 (b) A normal Helly circular-arc graph G_1

Figure 1: Normal Helly circular-arc model CM_1 and graph G_1

pose an algorithm that takes $O(n + m)$ time for the FVS problem in normal Helly circular-arc graphs.

The remainder of this paper is organized as follows. We state the definitions and notations used throughout this paper in Section 2. Next, we present our algorithm for the FVS problem and analyze its complexity in Section 3. Finally, we summarize our findings in Section 4 and conclude the paper by briefly discussing the scope for future work.

2 Definitions and Notations

In this section, we provide the definitions and relevant notations used throughout the paper. These establish the basis of the algorithm presented in Section 3.

2.1 Circular-arc Model and its Corresponding Graph

First, we provide the definitions of a circular-arc model and its corresponding graph. Consider a unit circle C and a family $\mathscr{F}$ of n arcs $A_1, A_2, \ldots, A_n$ along the circumference of C. Each arc A_i has two endpoints, a *left endpoint* a_i and a *right endpoint* b_i, and is denoted by $A_i = [a_i, b_i]$. The left endpoint a_i (resp., right endpoint b_i) is the last point encountered when traversing A_i counterclockwise (resp., clockwise). Without loss of generality, the coordinates of all left and right endpoints are distinct and are assigned consecutive integer values $1, 2, \ldots, 2n$ clockwise. Arc numbers i, j are assigned to each arc in increasing order of their right endpoints b_is, i.e., $A_i < A_j$ if $b_i < b_j$. The geometric representation described above is called a *circular-arc model* (CM). A graph $G = (V, E)$ is called a circular-arc graph if there exists a family of arcs $\mathscr{F} = \{A_1, A_2, \ldots, A_n\}$ such that there is a one-to-one correspondence between vertex $i \in V$ and $A_i \in \mathscr{F}$ such that an edge $(i, j) \in E$ if and only if A_i intersects with A_j in the *CM*. It is easy to see that a *CM* that fails to cover some points on the circle C is topologically the same as an *interval model*. Therefore, circular-arc graphs are a super-class of interval graphs.

2.2 Normal Helly circular-arc Model and its Corresponding Graph

A *normal Helly circular-arc model* is a circular-arc model where no the set of three arcs of $\mathscr{F}$ cover the entire circle C [26, 27]. A normal Helly circular-arc graph is an intersection graph corresponding to a normal Helly circular-arc model. Examples of a normal Helly circular-arc model and its corresponding graph are shown in Figure 1. Figure 1(a) shows a normal Helly circular-arc model CM_1 consisting of 12 arcs. For a normal Helly circular-arc model consisting of n arcs, an arc A_i with $b_n \in A_i$ and $i < n$ is called a *back-arc*. The set of all back-arcs is called the *back-arc set* and is denoted by BA. For CM_1, shown in Figure 1, we have a back-arc set $BA = \{A_1, A_2\}$ by $b_n = b_{12} = 22 \in A_1, A_2$. If a *CM* has no back-arc, it is topologically equivalent to an interval model. In this study, we assume that a *CM* has at least one back-arc. A normal Helly circular-arc graph is an intersection graph corresponding to a *CM*. Figure 1(b) shows the normal Helly circular-arc graph G_1 corresponding to CM_1.

Table 1: Example of $R(i)$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a_i	-5	-3	-1	0	3	5	10	11	8	13	16	18	19	21
b_i	1	2	4	6	7	9	12	14	15	10	20	22	25	26
$R(i)$	4	4	5	6	6	9	9	10	10	11	13	14	14	14

2.3 Maximal Clique Decomposition

A *maximal clique* is a clique to which no further vertices of a graph can be added such that it remains a clique. The maximal cliques of G_1, shown in Figure 1, are $MC_1 = \{1, 2, 3, 4\}$, $MC_2 = \{3, 4, 5\}$, $MC_3 = \{4, 5, 6\}$, $MC_4 = \{6, 9\}$, $MC_5 = \{7, 8, 9\}$, $MC_6 = \{8, 9, 10\}$, $MC_7 = \{10, 11\}$, $MC_8 = \{11, 12, 1\}$, and $MC_9 = \{12, 1, 2\}$. All maximal cliques can be generated in $O(n + m)$ time for a normal Helly circular-arc graph with n vertices and m edges [29, 28]. Let N_j be the cardinality of MC_j, i.e., $N_1 = 4$, $N_2 = 3$, $N_3 = 3$, $N_4 = 2$, $N_5 = 3$, $N_6 = 3$, $N_7 = 2$, $N_8 = 3$, and $N_9 = 3$. Let r be the number of maximal cliques of G_1. For each vertex i in G_1, $\sigma(i) = |\{MC_j \mid i \in MC_j, N_j \geqslant 3, 1 \leqslant j \leqslant r\}|$ and $\rho(i) = |\{MC_j \mid i \in MC_j, N_j = 2, 1 \leqslant j \leqslant r\}|$.

For graph G_1 shown in Figure 1, $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 2$, $\sigma(4) = 3$, $\sigma(5) = 2$, $\sigma(6) = 1$, $\sigma(7) = 1$, $\sigma(8) = 2$, $\sigma(9) = 2$, $\sigma(10) = 1$, $\sigma(11) = 1$, and $\sigma(12) = 2$. Moreover, $\rho(1) = 0$, $\rho(2) = 0$, $\rho(3) = 0$, $\rho(4) = 0$, $\rho(5) = 0$, $\rho(6) = 1$, $\rho(7) = 0$, $\rho(8) = 0$, $\rho(9) = 1$, $\rho(10) = 1$, $\rho(11) = 1$, and $\rho(12) = 0$. For the sake of convenience, we denote the σ and ρ value sequences of G_1 by $\sigma = [3, 2, 2, 3, 2, 1, 1, 2, 2, 1, 1, 2]$ and $\rho = [0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0]$, respectively.

2.4 Minimum Feedback Triangle-free Vertex Set

Throughout this paper, we use the term *triangle* to denote a cycle of length three. For a simple graph $G = (V, E)$, $F (\subset V)$ is a *feedback triangle-free vertex set* (FTS) if $G[V - F]$ has no triangle. In the example shown in Figure 1, $\{1, 4, 8\}$ or $\{1, 4, 9\}$ is a minimum cardinality FTS (MFTS) of G_1.

A *chordal graph* is a simple graph in which every cycle of length four or greater has a cycle chord. Interval graphs are a subclass of chordal graphs, i.e., they have no chordless cycle of length greater than three [18]. Hence, an MFTS is obviously an MFVS for interval graphs. On the other hand, normal Helly circular-arc graphs are a superclass of interval graphs and not a subclass of chordal graphs. They can have some chordless cycles of length greater than three. For example, the graph G_1 shown in Figure 1 has chordless cycles $\langle 1, 4, 6, 9, 10, 11, 1\rangle$ and $\langle 2, 3, 5, 6, 9, 10, 11, 12, 2\rangle$ of length six and eight, respectively. If F is an MFTS and not an MFVS of a normal Helly circular-arc graph G, $G[V - F]$ has a chordless cycle of length greater than three. A chordless cycle in $G[V - F]$ is called a *periphery*. For example, in Figure 1, $F = \{1, 4, 8\}$ is an MFTS of G_1, and $G_1[V - F]$ consists of a periphery $\langle 2, 3, 5, 6, 9, 10, 11, 12, 2\rangle$. Therefore, $F = \{1, 4, 8\}$ is not an MFVS, although F is an MFTS of G_1.

3 Algorithm and Its Correctness

In this section, we present our algorithm for solving the FVS problem for a normal Helly circular-arc graph.

3.1 Test Whether a CM is normal Helly

First, we must check whether a given circular-arc model CM is normal Helly. To do this, we employ an *extended circular-arc model* (ECM) constructed from CM. An ECM is constructed as follows. A given CM is cut at endpoint b_n (point 22 in CM_1 in Figure 1) on circumference C, and then, it is unrolled onto the real horizontal line. Moreover, each circular-arc $A_i = [a_i, b_i]$ in the CM is changed into a horizontal line segment $I_i = [a_i, b_i]$ called an *interval*. Here, each circular-arc $A_i = [a_i, b_i]$ with $a_i > b_i$ in the CM is replaced by two intervals $I_i = [a_i - 2n, b_i]$ and $I_{i+n} = [a_i, b_i + 2n]$. Note that I_{i+n} is a copy of I_i, and both I_i and I_{i+n} in the ECM correspond to the same circular-arc A_i in the CM. Figure 2 shows ECM_1 constructed from CM_1 shown in Figure 1.

Next, we define a function $R(i)$. $R(i)$ is the rightmost interval (including itself) that intersects interval I_i in an ECM, where if no interval intersects I_i then let $R(i)$ be i. Formally, $R(i) = \max\{\ k \mid b_i \in I_k, i \leqslant k \leqslant 2n\}$. All $R(i)$ are obtained in $O(n)$ time using prefix computation. Table 1 lists the details of $R(i)$ for ECM_1.

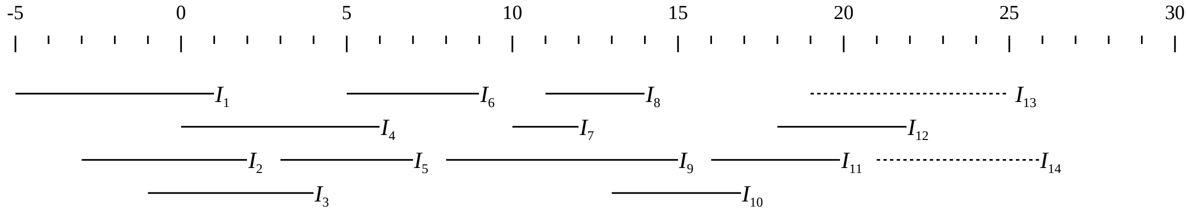
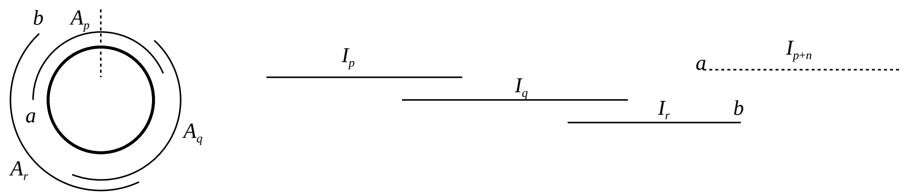
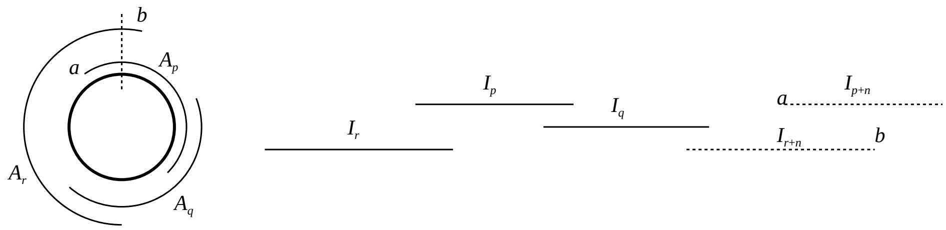


Figure 2: ECM_1 constructed from CM_1 in Figure 1



(a) A case where $A_p \in BA$, $q = R(p)$, and $r = R(q) = R(R(p))$.



(b) A case where $A_p, A_r \in BA$, $q = R(p)$, and $r + n = R(q) = R(R(p))$.

Figure 3: Illustration of three arcs A_p, A_q, and A_r that cover the entire circle.

We will use an ECM to check whether a given CM is normal Helly. By definition, if CM is a normal Helly circular-arc model, no the set of three arcs cover the entire circle C [26, 27].

Lemma 1 *Let ECM be an extended circular-arc model constructed from CM. If $b_{R(R(i))} < a_i + 2n$ for all $I_i \in BA$ in ECM, a given circular-arc model CM is normal Helly.*

Proof: By definition, if CM includes at least one set of three arcs that cover the entire circle C, it is not normal Helly. Suppose that CM is not normal Helly and contains the set of three arcs A_p, A_q, A_r that cover the entire circle C. One or two of these three arcs A_p, A_q, A_r must be back-arcs because if none of them is a back-arc, they cannot cover the entire circle. Moreover, if all of A_p, A_q, A_r are back-arcs, at least one triangle is constructed by them. Thus, when three arcs A_p, A_q, A_r cover the entire circle, one or two of them must be back-arcs.

Figure 3 shows examples of three arcs A_p, A_q, A_r covering the entire circle: ((a) is a case where $A_p \in BA$ and (b) is a case where $A_p, A_r \in BA$). $R(i)$ is the rightmost interval ($\geqslant i$) that intersects interval I_i in ECM. Figure 3(a) shows a case where $q = R(p)$ and $r = R(q) = R(R(p))$. If CM is not normal Helly, there exist the set of three arcs A_p, A_q, A_r that cover the entire circle in CM, i.e., $I_{R(R(p))}$ and I_{p+n} must intersect in ECM. Note that both of I_p and I_{p+n} in ECM correspond to the same circular-arc A_p in CM.

Thus, if $b_{R(R(i))} < a_i + 2n$ for all $A_i \in BA$ in ECM, a given circular-arc model CM is normal Helly. □

We present an algorithm (Algorithm 1) to check whether a given circular-arc model CM is normal Helly. The algorithm works as follows. In Step 1, an ECM is constructed from a given CM in $O(n)$ time. In Step 2, $R(i)$, $1 \leqslant i \leqslant n$, are computed. This step can be executed in $O(n)$ time using prefix computation. In Step 3, we check whether there are the set of three arcs that cover the entire circle. If there is no such set of three arcs, a given CM is normal Helly by Lemma 1. Step 3 can be executed in $O(n)$ time. Thus, Algorithm 1 can test whether a given circular-arc model CM is normal Helly in $O(n)$ time.

Algorithm 1: Check of a CM whether it is normal Helly or not

Input: The left and right points a_i, b_i of a CM.

(Step 1)
Construct an ECM from a CM.
(Step 2)
for *all* I_i, $1 \leqslant i \leqslant n$ **do** compute $R(i)$;
(Step 3)
if $b_{R(R(i))} < a_i + 2n$ *for all* $i \in BA$ **then**
| The CM is normal Helly;
else
| The CM is not normal Helly;
end

3.2 How to Compute an MFVS

In this section, we present an algorithm for solving the FVS problem for a normal Helly circular-arc graph. We will concisely describe the outline of our algorithm. First, we decompose a given normal Helly circular-arc graph into maximal cliques. An FTS is obtained by removing $N_j - 2$ vertices from each maximal clique MC_j. An MFTS is constructed by minimizing the number of removed vertices. At this point, if the constructed MFTS includes no periphery, it is an MFVS. Otherwise, we can obtain an MFVS by including a vertex for breaking the periphery in the MFTS.

Let $G = (V, E)$ be a normal Helly circular-arc graph corresponding to a *CM*. Algorithm 2 receives as an input the left and right points a_i, b_i of each A_i and back-arc set BA, and outputs an MFVS F of G. Now, we show how Algorithm 2 finds an MFVS of a given normal Helly circular-arc graph G. We use the graph G_1 shown in Figure 1 as an example to illustrate Algorithm 2 step by step (the updated part is underlined).

BEGIN

Step 1.
$MC_1 = \{1,2,3,4\}$, $MC_2 = \{3,4,5\}$, $MC_3 = \{4,5,6\}$, $MC_4 = \{6,9\}$, $MC_5 = \{7,8,9\}$, $MC_6 = \{8,9,10\}$, $MC_7 = \{10,11\}$, $MC_8 = \{11,12,1\}$, and $MC_9 = \{12,1,2\}$.

Step 2.
$\sigma = [\ 3,2,2,3,2,1,1,2,2,1,1,2\ ]$,
$\rho = [\ 0,0,0,0,0,1,0,0,1,1,1,0\ ]$.

Step 3.

Initialization
$U_1 = \{MC_1, MC_2, MC_3, MC_5, MC_6, MC_8, MC_9\}$,
$F = \emptyset$.

1st iteration

$[k = 1,\ U_2 = \{MC_1, MC_8, MC_9\}\ ]$
$MC_1 = \{1,2,3,4\}$, $F = \{\underline{1}, \underline{4}\}$,
$\sigma = [\ \underline{2}, \underline{1}, \underline{1}, \underline{2}, 2,1,1,2,2,1,1,2\ ]$.
$MC_8 = \{11,12,1\}$, $F = \{1,4\}$,
$\sigma = [\ \underline{1}, 1,1,2,2,1,1,2,2,1,\underline{0},\underline{1}\ ]$.
$MC_9 = \{12,1,2\}$, $F = \{1,4\}$,
$\sigma = [\ \underline{0},\underline{0},1,2,2,1,1,2,2,1,0,\underline{0}\ ]$.
$U_1 = \{MC_2, MC_3, MC_5, MC_6\}$.

2nd iteration

$[k = 4,\ U_2 = \{MC_2, MC_3\}\ ]$
$MC_2 = \{3,4,5\}$, $F = \{1,4\}$,
$\sigma = [\ 0,0,\underline{0},\underline{1},\underline{1},1,1,2,2,1,0,0\ ]$.
$MC_3 = \{4,5,6\}$, $F = \{1,4\}$,
$\sigma = [\ 0,0,0,\underline{0},\underline{0},\underline{0},1,2,2,1,0,0\ ]$.
$U_1 = \{MC_5, MC_6\}$.

3rd iteration

$[k = 8,\ U_2 = \{MC_5, MC_6\}\ ]$
$MC_5 = \{7,8,9\}$, $F = \{1,4,\underline{9}\}$,
$\sigma = [\ 0,0,0,0,0,0,\underline{0},\underline{1},\underline{1},1,0,0\ ]$,
$\rho = [\ 0,0,0,0,0,1,0,0,1,1,1,0\ ]$.
$MC_6 = \{8,9,10\}$, $F = \{1,4,9\}$,
$\sigma = [\ 0,0,0,0,0,0,0,\underline{0},\underline{0},\underline{0},0,0\ ]$.
$U_1 = \emptyset$.

Step 4.
$F = \{1,4,9\}$ is an MFVS because $G[V - F]$ has no periphery.

END

Algorithm 2: Algorithm for obtaining MFVS

Input: The left and right points a_i, b_i of each circular-arc $A_i, i = 1, 2, \ldots, n$ and a back-arc set BA.
Output: The minimum feedback vertex set F.

(Step 1)
Compute all maximal cliques $MC_j,\ j = 1, 2, \ldots, r$;
Let r be the total number of maximal cliques of G ;

(Step 2)
Compute all $\sigma(i)$ for $i = 1, 2, \ldots, n$;
Compute all $\rho(i)$ for $i = 1, 2, \ldots, n$;

(Step 3) /* Obtain MFTS */
Initialization ;
$U_1 := \{MC_j \mid N_j \geqslant 3, j = 1, 2, \ldots, r\}$;
$F := \emptyset$;
while $U_1 \neq \emptyset$ **do**
  $k := \arg\max \sigma(i)$;
  $U_2 := \{MC_j \mid k \in MC_j, MC_j \in U_1\}$;
  for *all* $MC_j \in U_2$ **do**
    if $|MC_j - F| \geqslant 3$ **then**
      Select $|MC_j - F| - 2$ vertices in lexicographical order with respect to $(\sigma(i), \rho(i))$ where $(\sigma(i), \rho(i)) > (\sigma(j), \rho(j))$ if $\sigma(i) > \sigma(j)$ or $\sigma(i) = \sigma(j),\ \rho(i) > \rho(j)$;
      Add the selected $|MC_j - F| - 2$ vertices to F ;
    end
    for $i \in MC_j$ **do** $\sigma(i) := \sigma(i) - 1$;
  end
  $U_1 := U_1 - U_2$;
end

(Step 4) /* Obtain MFVS */
if $G[V - F]$ *has a periphery* **then**
  if $|BA - F| = 1$ **then**
    Add vertex k ($BA - F = \{k\}$) to F ;
    Output F ;
  end
  if $|BA - F| = 2$ **then**
    We select a vertex k such that $a_k = \min\{a_{k_1}, a_{k_2}\}$ ($BA - F = \{k_1, k_2\}$) ;
    Add vertex k to F ;
    Output F ;
  end
else
  Output F ;
end

In Step 1, we compute all maximal cliques MC_j, $1 \leqslant j \leqslant r$, for G_1 by applying the algorithm presented in [29, 28]. In Step 2, for all $1 \leqslant i \leqslant n$, we compute $\sigma(i) = |\{MC_j \mid i \in MC_j, N_j \geqslant 3, 1 \leqslant j \leqslant r\}|$ and $\rho(i) = |\{MC_j \mid i \in MC_j, N_j = 2, 1 \leqslant j \leqslant r\}|$. In Step 3, we efficiently compute an MFTS of G_1. A graph obtained by deleting all but two vertices from each maximal clique $MC_j(N_j \geqslant 3)$, $1 \leqslant j \leqslant r$, has no triangle. Step 3 constructs an MFTS by adding $N_j - 2$ vertices of each $MC_j(N_j \geqslant 3)$ to F. In this example, Step 3 is executed in seven iterations because there are seven maximal cliques with $N_j \geqslant 3$. After executing Step 3, we obtain $F = \{1, 4, 9\}$ as an MFTS of G_1. In Step 4, we obtain $F = \{1, 4, 9\}$ as an MFVS of G_1 because $G[V - F]$ has no periphery.

Lemma 2 shows that F is an MFTS of G following the execution of Step 3 of Algorithm 2.

Lemma 2 *Let G be a normal Helly circular-arc graph. Following the execution of Step 3, F is an*

MFTS of G.

Proof: Each triangle contained in G is a subset of any maximal clique $MC_j(N_j \geqslant 3)$ in G. A graph obtained by deleting all but two vertices from each maximal clique $MC_j(N_j \geqslant 3)$, $1 \leqslant j \leqslant r$, has no triangle. Thus, a set F consisting of $N_j - 2$ vertices of each $MC_j(N_j \geqslant 3)$, $1 \leqslant j \leqslant r$, is an FTS of G. It is obvious that the cardinality of F can be reduced by including vertices that appear in many maximal cliques in F. By definition, $\sigma(i) = |\{MC_j \mid i \in MC_j, N_j \geqslant 3, 1 \leqslant j \leqslant r\}|$. In order to obtain an MFTS, we construct F by selecting $N_j - 2$ vertices in decreasing order of σ.

In Step 3, initially, we set $U_1 = \{MC_j \mid N_j \geqslant 3, j = 1, 2, \ldots, r\}$ and $F = \emptyset$. Next, we compute $k = \arg\max \sigma(i)$ and set $U_2 = \{MC_j \mid k \in MC_j, MC_j \in U_1\}$, i.e., vertex k is contained in the largest number of maximal cliques, and U_2 is the set of maximal cliques containing k. Therefore, we can reduce the cardinality of F by breaking the triangles in $MC_j \in U_2$ by priority.

In the first iteration, we select all vertices except two minima with σ values in MC_j and add them to F for each $MC_j \in U_2$. Here, assume that U_2 consists of m maximal cliques, i.e., $U_2 = \{MC_1, MC_2, \ldots, MC_m\}$. Then, a subgraph $G[MC_1 - F]$ has no triangle and F is clearly an MFTS of $G[MC_1]$. In the next step, if $|MC_2 - F| < 3$, no vertex is added to F. This implies that the elimination of vertices in F obtained in the previous step breaks all triangles of MC_2. If $|MC_2 - F| \geq 3$, we select $|MC_2 - F| - 2$ vertices in decreasing order of σ in $MC_2 - F$ and add them to F. It is obvious that the cardinality of F can be reduced by adding vertices that appear in several maximal cliques in F. Following this step, $G[MC_1 \cup MC_2 - F]$ has no triangle and F is an MFTS of $G[MC_1 \cup MC_2]$. Similarly, in the next step, if $|MC_3 - F| \geq 3$, we select $|MC_3 - F| - 2$ vertices in decreasing order of σ in $MC_3 - F$ and add them to F. $G[MC_1 \cup MC_2 \cup MC_3 - F]$ has no triangle, and F is an MFTS of $G[MC_1 \cup MC_2 \cup MC_3]$. Using a similar argument, following the execution of the m-th step, $G[MC_1 \cup MC_2 \cup \cdots \cup MC_m - F]$ has no triangle, and F is an MFTS of $G[MC_1 \cup MC_2 \cup \cdots \cup MC_m]$.

In the second iteration, we update U_1 to be $U_1 - U_2$ and calculate $k = \arg\max \sigma(i)$. As in the case of the first iteration, we select all vertices except two minima with σ values in MC_j and add them to F for each $MC_j \in U_2$. Step 3 of Algorithm 2 repeats the processes described above until U_1 becomes an empty set. The method described above thus constructs an MFTS F of the normal Helly circular-arc graph G. □

Lemma 3 *Let G be a normal Helly circular-arc graph and let F be an MFTS of G. After executing Step 3, if $G[V - F]$ has no periphery, F is an MFVS of G.*

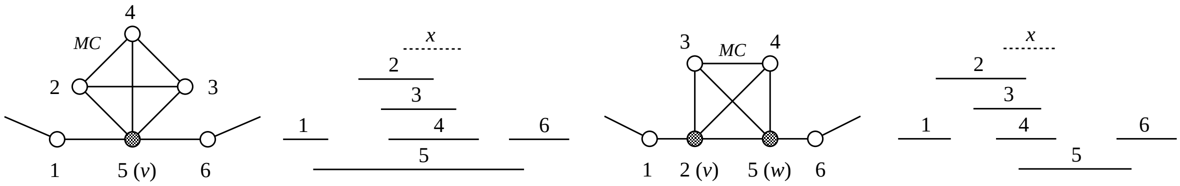
Proof: Because F is an MFTS of G, $G[V - F]$ has no triangle. If $G[V - F]$ has neither a triangle nor a periphery, $G[V - F]$ has no cycle. Thus, such F is an MFVS of G. □

Here, we explain how $\rho(i)$ are used to find an MFTS in Step 3 of Algorithm 2. In the example shown in Figure 1, both vertex sets $\{1, 4, 8\}$ and $\{1, 4, 9\}$ are MFTSs of G_1. In general, not all MTFSs of G_1 are MFVSs of G_1. For example, a set $\{1, 4, 9\}$ is an MFVS of G_1; however, $\{1, 4, 8\}$ is not an MFVS because a subgraph $G_1[V - \{1, 4, 8\}]$ has a periphery $\langle 2, 3, 5, 6, 9, 10, 11, 12, 2\rangle$. We describe how Step 3 of Algorithm 2 constructs an MFTS F such that $G[V - F]$ has no periphery, if possible.

Consider the case where a maximal clique $MC_j(N_j \geqslant 3)$ and a periphery have a vertex v (5 in Figure 4(a)) in common. In Step 3, we select $|MC_j - F| - 2$ vertices from MC_j to break all triangles in MC_j and include them in F. Here, if the vertex v is in F, $G[V - F]$ has neither a triangle nor a periphery. A vertex v containing MC_j and a periphery in common must be included in some $MC_k(N_k = 2)$ ($\{1, 5\}$ or $\{5, 6\}$ in Figure 4(a)). Moreover, we have no maximal clique $MC_k(N_k = 2)$ containing vertices u (2, 3, 4) except v in MC_j. This is because it is clear from the corresponding model that if there exists a vertex x adjacent to u $(\neq v)$ in MC_j, G must have a triangle $\langle uvx\rangle$. Therefore, we select $|MC_j - F| - 2$ vertices in lexicographical order with respect to $(\sigma(i), \rho(i))$ where $(\sigma(i), \rho(i)) > (\sigma(j), \rho(j))$ if $\sigma(i) > \sigma(j)$ or $\sigma(i) = \sigma(j)$, $\rho(i) > \rho(j)$.

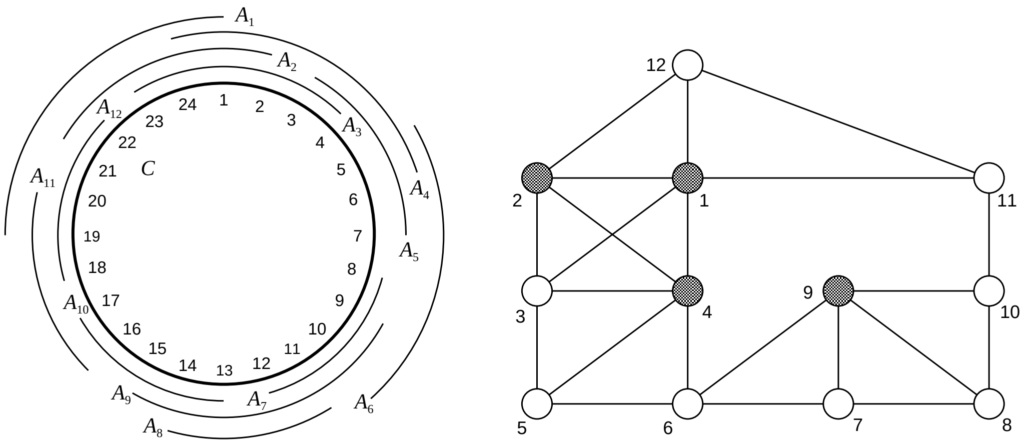
Next, we consider the case of a maximal clique $MC_j(N_j \geqslant 3)$ and a periphery with two vertices v, w (2 and 5 in Figure 4(b)) in common. It is obvious that a periphery in G is broken by removing either v or w from G. In this case, for v and w, there exist maximal cliques $MC_k(N_k = 2)$ ($\{1, 2\}$ and $\{5, 6\}$ in Figure 4(b)) containing v and w, respectively. Moreover, we have no maximal clique $MC_k(N_k = 2)$ containing vertex u, except v and w (3, 4 in Figure 4(b)) in MC_j. This is because it is clear from the corresponding model that if there exists a vertex x adjacent to $u(\neq v)$ in MC_k, G must have a triangle $\langle uvx\rangle$ or $\langle uwx\rangle$. Therefore, we select $|MC_j - F| - 2$ vertices in lexicographical order with respect to $(\sigma(i), \rho(i))$ where $(\sigma(i), \rho(i)) > (\sigma(j), \rho(j))$ if $\sigma(i) > \sigma(j)$ or $\sigma(i) = \sigma(j)$, $\rho(i) > \rho(j)$.

By executing the method described above, Step 3 outputs an MFTS F such that a normal Helly circular-arc graph $G[V - F]$ has neither a triangle nor a periphery, if possible.



(a) MC shares a vertex with a periphery (b) MC shares two vertices with a periphery

Figure 4: A maximal clique and a periphery sharing vertices



(a) A normal Helly circular-arc model CM_2 (b) A normal Helly circular-arc graph G_2

Figure 5: Normal Helly circular-arc model CM_2 and graph G_2

In the third iteration of the execution example shown in Figure 1, we can add either vertex 8 or vertex 9 into F for $MC_5 = \{7, 8, 9\}$ because $\sigma(7) = 0$ and $\sigma(8) = \sigma(9) = 1$. Therefore, we add vertex 9 into F because $\rho(8) = 0$ and $\rho(9) = 1$. $F = \{1, 4, 9\}$, which we obtained after executing Step 3, is an MFTS of G_1 by Lemma 2. Moreover, by Lemma 3, $F = \{1, 4, 9\}$ is an MFVS because $G_1[V - F]$ has no periphery.

Thus far, we have presented an example where an MFTS of a normal Helly circular-arc graph G is also an MFVS of G. However, there exist cases where an MFTS of G obtained by executing Step 3 is not an MFVS of G. The graph G_2 shown in Figure 5 is a normal Helly circular-arc graph. Next, we describe the procedure to construct an MFTS of G_2 shown in Figure 5 by executing Step 3.

BEGIN

Step 1.

$MC_1 = \{1, 2, 3, 4\}$, $MC_2 = \{3, 4, 5\}$, $MC_3 = \{4, 5, 6\}$, $MC_4 = \{6, 7, 9\}$, $MC_5 = \{7, 8, 9\}$, $MC_6 = \{8, 9, 10\}$, $MC_7 = \{10, 11\}$, $MC_8 = \{11, 12, 1\}$, and $MC_9 = \{12, 1, 2\}$.

Step 2.

$\sigma = [\ 3, 2, 2, 3, 2, 2, 2, 2, 3, 1, 1, 2\ ]$,
$\rho = [\ 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0\ ]$.

Step 3.

Initialization

$U_1 = \{MC_1, MC_2, MC_3, MC_4, MC_5, MC_6, MC_8, MC_9\}$,
$F = \emptyset$.

1st iteration

[$k = 1$, $U_2 = \{MC_1, MC_8, MC_9\}$]
$MC_1 = \{1, 2, 3, 4\}$, $F = \{\underline{1}, \underline{4}\}$,
$\sigma = [\ \underline{2}, \underline{1}, \underline{1}, \underline{2}, 2, 2, 2, 2, 3, 1, 1, 2\ ]$.
$MC_8 = \{11, 12, 1\}$, $F = \{1, 4\}$,
$\sigma = [\ \underline{1}, 1, 1, 2, 2, 2, 2, 2, 3, 1, \underline{0}, \underline{1}\ ]$.
$MC_9 = \{12, 1, 2\}$, $F = \{1, 4\}$,
$\sigma = [\ \underline{0}, \underline{0}, 1, 2, 2, 2, 2, 2, 3, 1, 0, \underline{0}\ ]$.
$U_1 = \{MC_2, MC_3, MC_4, MC_5, MC_6\}$.

2nd iteration

[$k = 9$, $U_2 = \{MC_4, MC_5, MC_6\}$]
$MC_4 = \{6, 7, 9\}$, $F = \{1, 4, \underline{9}\}$,
$\sigma = [\ 0, 0, 1, 2, 2, \underline{1}, \underline{1}, 2, \underline{2}, 1, 0, 0\ ]$.
$MC_5 = \{7, 8, 9\}$, $F = \{1, 4, 9\}$,
$\sigma = [\ 0, 0, 1, 2, 2, 1, \underline{0}, \underline{1}, \underline{1}, 1, 0, 0\ ]$.
$MC_6 = \{8, 9, 10\}$, $F = \{1, 4, 9\}$,
$\sigma = [\ 0, 0, 1, 2, 2, 1, 0, \underline{0}, \underline{0}, \underline{0}, 0, 0\ ]$.
$U_1 = \{MC_2, MC_3\}$.

3rd iteration

$[k = 4,\ U_2 = \{MC_2, MC_3\}\ ]$

$MC_2 = \{3, 4, 5\}$, $F = \{1, 4, 9\}$,

$\sigma = [\ 0, 0, \underline{0}, \underline{1}, \underline{1}, 1, 0, 0, 0, 0, 0, 0\ ]$.

$MC_3 = \{4, 5, 6\}$, $F = \{1, 4, 9\}$,

$\sigma = [\ 0, 0, 0, \underline{0}, \underline{0}, \underline{0}, 0, 0, 0, 0, 0, 0\ ]$.

$U_1 = \emptyset$.

END

After executing Step 3, $F = \{1, 4, 9\}$ constructed in this example is an MFTS of G_2. Removing all vertices in $F = \{1, 4, 9\}$ breaks all triangles in G_2. However, F is not an MFVS of G_2 because $G_2[V - F]$ consists of a periphery $\langle 2, 3, 5, 6, 7, 8, 10, 11, 12, 2\rangle$. The periphery remains unbroken because it does not contain any of $F = \{1, 4, 9\}$. In fact, there exists no MFVS of cardinality three in G_2. Hence, we can obtain an MFVS by including a vertex for breaking the periphery in F if $G[V - F]$ consists of a periphery.

Lemma 4 *Let G be a normal Helly circular-arc graph. If F is an MFTS and not an MFVS of G, $G[V - F]$ has a periphery that must contain one or two back-arcs.*

Proof: $G[V - F]$ has no triangle because F is an MFTS of G. Moreover, F is not an MFVS of G. Thus, $G[V - F]$ must have a chordless cycle of length greater than three, i.e., a periphery.

As mentioned in Section 2, interval graphs are a subclass of normal Helly circular-arc graphs and have no periphery. A normal Helly circular-arc model from which all back-arcs are removed is topologically equivalent to an interval model. Therefore, $G[V - F]$ must contain at least one back-arc because it has a periphery.

$G[V - F]$ does not contain three or more back-arcs. If $G[V - F]$ has three back-arcs, these three back-arcs cover a point (b_n) on the circumference of a circle C by the definition of a back-arc. This implies that $G[V - F]$ contains a triangle. Thus, it contradicts the proposition that F is an MFTS of G.

Thus, if F is an MFTS and not an MFVS of G, $G[V - F]$ has a periphery that must contain one or two back-arcs. □

Lemma 5 *Let G be a normal Helly circular-arc graph. Following the execution of Step 4 of Algorithm 2, F is an MFVS of G.*

Proof: By Lemma 4, if $G[V - F]$ consists of a periphery, such a periphery must contain one or two back-arcs.

In the case where $G[V - F]$ consists of a periphery and contains one back-arc A_1, we can break the periphery by removing A_1 (Figure 6(a)). Thus, we can obtain an MFVS by adding A_1 to F.

We consider the cases where $G[V - F]$ consists of a periphery and contains two back-arcs ($A_1 < A_2$). Because both back-arcs cover point b_n by definition, these back-arcs must intersect. There are two possible cases when $G[V - F]$ contains two back-arcs. The first satisfies $a_1 < a_2$ (Figure 6(b)), and the second satisfies $a_1 > a_2$ (Figure 6(c)). For the former, the periphery is broken by removing A_1. For the latter, the periphery is broken by removing A_2. Therefore, we can break the periphery by removing a back-arc A_k such that $a_k = \min\{a_1, a_2\}$.

Thus, we can construct an MFVS after executing Step 4. □

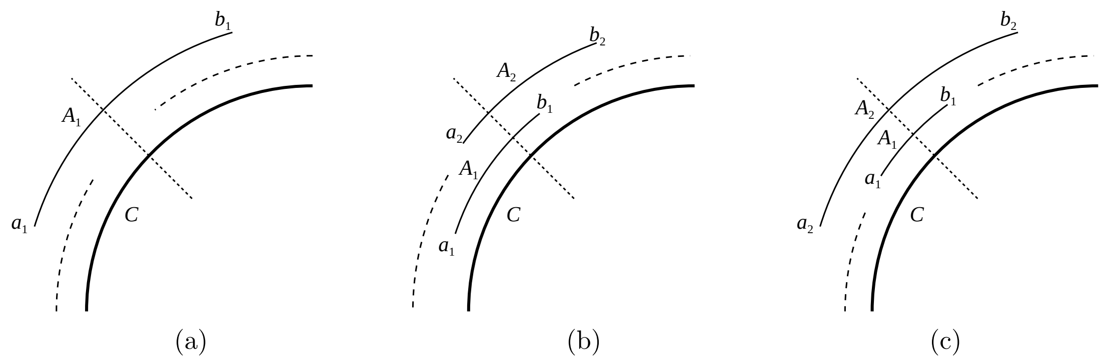


Figure 6: Illustration of Lemma 5

In Step 3, we obtain $F = \{1, 4, 9\}$ in the example shown in Figure 5. In Step 4, $G[V - F]$ has a periphery $\langle 2, 3, 5, 6, 7, 8, 10, 11, 12, 2 \rangle$. We include vertex 2 in F by $|BA - F| = 1$ and $BA - F = \{2\}$. Therefore, we have $F = \{1, 2, 4, 9\}$, and thus, F is an MFVS of G_2.

Next, we analyze the complexity of Algorithm 2. In Step 1, all maximal cliques of G are computed in $O(n + m)$ time [29, 28]. In Step 2, $\sigma(i)$ and $\rho(i)$ are computed for all $i \in V$. The complexity of this step depends on the number of maximal cliques of G, which is at most the number of vertices of G [28]. Thus, Step 2 can be executed in $O(n)$ time. In Step 3, an MFVS F of G is constructed. This step requires as many iterations as the number of maximal cliques. Thus, Step 3 is executed in $O(n)$ time. In Step 4, we check whether $G[V - F]$ consists of a periphery. If $G[V - F]$ consists of a periphery, one vertex of $BA - F$ is added to F. This step is executed in $O(n)$ time. Thus, we have the following theorem.

Theorem 1 *Given a normal Helly circular-arc graph G, Algorithm 2 finds an MFVS of G in $O(n+m)$ time.*

4 Concluding Remarks

In this study, we proposed an algorithm that takes $O(n + m)$ time to find an MFVS on a normal Helly circular-arc graph with n vertices and m edges. Our algorithm employs algorithms to find maximal cliques [29, 28] according to a method that can be understood intuitively. The complexity of our algorithm depends on the number of maximal cliques in a normal Helly circular-arc graph. Reducing the complexity of the algorithm and extending the results to other graphs are issues to be considered in future research.

Acknowledgments

This work was partially supported by JSPS KAKENHI Grant Number 25330019 and 26330359.

References

[1] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, Volume 57, Second Edition, Elsevier, 2004.

[2] F.W. Stahl, "Circular genetic maps," J. Cell. Physiol., vol.70, pp.1–12, 1967.

[3] L. Hubert, "Some applications of graph theory and related non-metric techniques to problems of approximate seriation: the case of symmetric proximity measures," British. J. Math. Statist. Psych., vol.27, pp.133–153, 1974.

[4] A. Tucker, "Coloring a family of circular-arcs," SIAM J. Applied Mathematics, vol.29, no.3, pp.493–502, 1975.

[5] S. Stefanakos and T. Erlebach, "Routing in all-optical ring networks revisited," Proc. 9th IEEE Symposium on Computers and Communication, pp.288–293, 2004.

[6] M.C. Lin, F.J. Soulignac, and J.L. Szwarcfiter, "The clique operator on circular-arc graphs," Discrete Appl. Math., vol.158, no.12, pp.1259–1267, 2010.

[7] D. Bhowmick and L.S. Chandran, "Boxicity of circular arc graphs," Graphs and Combinatorics, vol.27, no.6, pp.769–783, 2011.

[8] H. Kaplan and Y. Nussbaum, "A simpler linear-time recognition of circular-arc graphs," Algorithmica, vol.61, no.3, pp.694–737, 2011.

[9] M.R. Cerioli and A.L. Korenchendler, "Clique-coloring circular-arc graphs," Electronic Notes in Discrete Mathematics, vol.35, pp.287–292, 2009.

[10] D. Jungnickel, Graphs, Networks and Algorithms, Springer, 2000.

[11] A. Silberschatz, P.B. Galvin, and G. Gagne, Operating Systems Concepts, John Wiley and Sons, New York, 2003.

[12] D.B. Johnson, "Finding all the elementary circuits of a directed graph," SIAM J. Comput., vol.4, no.1, pp.77–84, 1975.

[13] A.V. Hudli and R.V. Hudli, "Finding small feedback vertex sets for VLSI circuits," Microprocessors and Microsystems, vol.18, no.7, pp.393–400, 1994.

[14] D. Gusfield, "A graph theoretic approach to statistical data security," SIAM J. Comput., vol.17, no.3, pp.552–571, 1998.

[15] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to Theory of NP-completeness, W. H. Freeman, San Francisco, 1979.

[16] M. Yannakakis, "Node-deletion problem on bipartite graphs," SIAM J. Comput., vol.10, no.2, pp.310–327, 1981.

[17] C.L. Lu and C.Y. Tang, "A linear-time algorithm for the weighted feedback vertex problem on interval graphs," Inf. Process. Lett., vol.61, no.2, pp.107–111, 1997.

[18] A. Saha and M. Pal, "An algorithm to find a minimum feedback vertex set of an interval graph," Advanced Modeling and Optimization, vol.7, no.1, pp.99–116, 2005.

[19] Y.D. Liang, "On the feedback vertex set problem in permutation graphs," Inf. Process. Lett., vol.52, no.3, pp.123–129, 1994.

[20] F.L. Luccio, "Almost exact minimum feedback vertex set in meshes and butterflies," Inf. Process. Lett., vol.66, no.2, pp.59–64, 1998.

[21] R. Focardi, F.L. Luccio, and D. Peleg, "Feedback vertex set in hypercubes," Inf. Process. Lett., vol.76, no.1-2, pp.1–5, 2000.

[22] F.H. Wang, Y.L. Wang, and J.M. Chang, "Feedback vertex sets in star graphs," Inf. Process. Lett., vol.89, no.4, pp.203–208, 2004.

[23] F. Carrabs, R. Cerulli, M. Gentili, and G. Parlato, "A linear time algorithm for the minimum weighted feedback vertex set on diamonds," Inf. Process. Lett., vol.94, no.1, pp.29–35, 2005.

[24] C.J. Kuo, C.C. Hsu, H.R. Lin, and K.K. Lin, "An efficient algorithm for minimum feedback vertex sets in rotator graphs," Inf. Process. Lett., vol.109, no.9, pp.450–453, 2009.

[25] A. Tucker, "An efficient test for circular-arc graphs," SIAM J. Comput., vol.9, no.1, pp.1–24, 1980.

[26] T.A. McKee, "Restricted circular-arc graphs and clique cycles," Discrete Math., vol.263, no.1-3, pp.221–231, 2003.

[27] M.C. Lin, F.J. Soulignac, and J.L. Szwarcfiter, "Normal Helly circular-arc graphs and its subclasses," Discrete Appl. Math., vol.161, no.7-8, pp.1037–1059, 2013.

[28] G. Caimi, M. Fuchsberger, and R. Zenklusen, "About maximal cliques in a certain class of circular-arc graphs," working paper, eth zurich, Institute for Operation Research, 2008.

[29] M. Pal and G.P. Bhattacharjee, "An optimal parallel algorithm for computing all maximal cliques of an interval graph and its applications," J. Institution of Engineers (India), vol.76, pp.22–33, 1995.