The heat kernel method for the tempered distributions on the Heisenberg group

Yasuyuki OKA
Department of Creative Engineering, National Institute of Technology, Kushiro College

Abstract. In [9], J. Kim and M. W. Wong gave the heat kernel method for the tempered distributions on the Heisenberg group. However, they used some propositions without the proofs. In this paper, we will introduce the heat kernel method for the tempered distributions on the Heisenberg group while making up for their deficiency and our recent results for the heat kernel method in [12] and [13].
Keyword: Heisenberg group, Tempered distributions, Heat kernel method, Heat equation, Schwartz kernel theorem.

1 Introduction

The Eucleadian space is the simplest and typical example of the Riemannian manifold. In 1987, the heat kernel method was given by T. Matsuzawa in [11] to the space of the hyperfunctions with a compact support on the Eucleadian space and several mathematicians gave the heat kernel method for the several functional spaces on the Eucleadian space, for example, the space of the Fourier hyperfunctions, the space of the tempered distributions, the space of the distributions of exponential growth, the dual space of the Gel'fand-Shilov space and etc.

On the other hand, The Heisenberg group is the simplest and typical example of the sub-Riemannian manifold and the most nearly Eucleadian space in the non commutative geometry. Moreover it is the step 2 sub-Riemannian manifold. The sub-Riemannian manifold may be interpreted as a generalization of the Riemannian manifold. The difference is that the motion for the sub-Riemannian manifold is restricted to the horizontal direction (for the Riemannian manifold, we can measure the velocity and distance in all directions). The Heisenberg group has been investigated by many mathematicians. In 2006, J. Kim and M. W. Wong gave the heat kernel method for the tempered distributions on the Heisenberg group. By this result, we can start the investigation of the differential equations associated with the Heisenberg group using the heat kernel method. They obtained the magnificent result. However, they showed the heat kernel method using some propositions without the proofs.

In this paper, we will mainly introduce the heat kernel method for the tempered distributions on the Heisenberg group while making up for their deficiency and give our recent results for the heat kernel method in [12] and [13].

2 The Heisenberg group $\mathbb{H}^d$

First of all, we fix some notations. We use a multi-index $\beta \in \mathbb{Z}_+^d$, namely, $\beta = (\beta_1, \cdots, \beta_d)$, where $\beta_i \in \mathbb{Z}$ and $\beta_i \geq 0$. So, for $x \in \mathbb{R}^d$, $x^\beta = x_1^{\beta_1} \cdots x_d^{\beta_d}$ and $\partial_x^\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_d}^{\beta_d}$, where $\partial_{x_j}^{\beta_j} = (\partial/\partial x_j)^{\beta_j}$. Moreover $\Delta = \sum_{j=1}^d \partial^2/\partial x_j^2$.

We recall the definition and the properties of the Heisenberg group. We refer to [1], [2], [3], [6], [15] and [16]. Let $g = (x, y, t)$ and $g' = (x', y', t') \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} = \mathbb{R}^{2d+1}$. Then we define the group law of $\mathbb{R}^{2d+1}$ by

$$\begin{aligned}&(x, y, t)(x', y', t')\\ &= (x + x', y + y', t + t' + 2(x' \cdot y - x \cdot y')),\end{aligned} \tag{2.1}$$

where $x \cdot y = \sum_{j=1}^d x_j y_j$. The group $\mathbb{R}^{2d+1}$ with respect to the group law defined by (2.1) is called the Heisenberg group and denoted by $\mathbb{H}^d$. Its identity element is $\boldsymbol{e} = (0, 0, 0)$ and the inverse of the element (x, y, t) is $(x, y, t)^{-1} = (-x, -y, -t)$. The Heisenberg group $\mathbb{H}^d$ is a locally compact Hausdorff group and its Haar measure is the Lebesgue measure $dxdydt$. The left-invariant vector fields in the Heisenberg group $\mathbb{H}^d$ as $\mathbb{R}^{2d+1}$ are represented by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \ X_{d+j} = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}$$

and

$$X_{2d+1} = \frac{\partial}{\partial t}$$

for $j = 1, 2, \cdots, d$ and these make a basis for the Lie algebra of $\mathbb{H}^d$. Since their first brackets

$$[X_j, X_{d+j}] = -4X_{2d+1},$$

the induced geometry is step 2. The sub-Laplacian $\Delta_{\mathbb{H}^d}$ on $\mathbb{H}^d$ is defined by $\Delta_{\mathbb{H}^d} = \sum_{j=1}^{2d} X_j^2$. We consider the heat operator

$$\partial/\partial s - \Delta_{\mathbb{H}^d}$$

on $\mathbb{H}^d \times (0, \infty)$.

Let $\lambda > 0$. Then we define the dilations δ_λ by $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$ for $(x, y, t) \in \mathbb{H}^d$. The homogeneous dimension Q of $\mathbb{H}^d$ is given by $Q = 2d + 2$. Moreover, a function u from $\mathbb{H}^d$ to $\mathbb{C}$ is called the Heisenberg-homogeneous of degree $k \in \mathbb{Z}$ if $u \circ \delta_\lambda = \lambda^k u$ for $\lambda > 0$. Especially the Heisenberg-homogeneous of degree of the distance function ρ defined by $\rho(g) = ((x^2 + y^2)^2 + t^2)^{\frac{1}{4}}$ for $g = (x, y, t) \in \mathbb{H}^d$ is one, that is, $\rho(\lambda x, \lambda y, \lambda^2 t) = \lambda\rho(x, y, t)$. The following estimate also holds:

$$\rho(g'^{-1}g) \leq \rho(g) + \rho(g').$$

The distance between two points g and g' in $\mathbb{H}^d$ is given by $d_K(g, g') := \rho\left(g'^{-1}g\right)$. Especially, we denote by $d_K(g)$ the distance from the origin. This distance function ρ is called Korányi norm and the distance d_K is called Korányi distance.

The horizontal distribution is defined by $\mathcal{I}_g = \mathrm{span}_g\{X_1, X_2, \cdots, X_{2d}\}$. Then we shall consider the non-degenerate, positive definite bilinear form $\langle\cdot,\cdot\rangle : \mathcal{I}_g \times \mathcal{I}_g \to \mathbb{R}$ at any point $g \in \mathbb{H}^d$ such that $\langle X_i, X_j\rangle = \delta_{i,j}$ $(i, j = 1, 2, \cdots, 2d)$, where $\delta_{i,j}$ means Kronecker's delta. The length $l(\gamma)$ of the horizontal curve $\gamma(t)$, $t \in [a, b]$ is defined by

$$l(\gamma) = \int_a^b \langle\dot\gamma(t), \dot\gamma(t)\rangle\, dt.$$

The Carnot-Carathéodry distance $d_{CC}(g, g')$ between two points $g, g' \in \mathbb{H}^d$ is defined by the infimum of the lengths of all smooth horizontal curves joining g to g' (see [1]). These distances are bi-Lipschitz equivalent. Thus, there exists a constant $C_1 > 1$ such that

$$\frac{1}{C_1} d_{CC} \leq d_K \leq C_1 d_{CC}$$

(for instance, see [10]).

Let f and h be suitable functions on $\mathbb{H}^d$. Then we define the convolution $f * h$ of f with h as follows:

$$(f * h)(g) = \int_{\mathbb{H}^d} f(g')h(g'^{-1}g)dg'$$

for $g, g' \in \mathbb{H}^d$. The convolution on $\mathbb{H}^d$ is non-commutative, in general.

Finally, we give the following uniquness theorem.

Proposition 1 ([14])**.** *Let $U_s(g)$ be a solution to the Cauchy problem*

$$\begin{cases} \dfrac{\partial}{\partial s} U_s(g) = \Delta_{\mathbb{H}^d} U_s(g), \\ U_0(g) = 0 \end{cases}$$

in $\mathbb{H}^d \times (0, S)$ and be a continuous function in $\mathbb{H}^d \times [0, S]$ satisfying the condition: There exists a constant $C > 0$ such that

$$|U_s(g)| \leq Ce^{ad_K(g)^2} \tag{2.2}$$

for some constant $a > 0$. Then $U \equiv 0$.

3 The space $\mathcal{S}(\mathbb{H}^d)$ and its dual space $\mathcal{S}'(\mathbb{H}^d)$

Let $k \in \mathbb{N}$ and a multi-index

$$\alpha \in \{0, 1, 2, \cdots, 2d\}^k$$
$$= \underbrace{\{0, 1, 2, \cdots, 2d\} \times \cdots \times \{0, 1, 2, \cdots, 2d\}}_{k}.$$

Then the functions $(X_\alpha\varphi)(g)$ are defined by

$$(X_\alpha\varphi)(g) = (X_{\alpha_1}X_{\alpha_2}\cdots X_{\alpha_k}\varphi)(g)$$

for a function $\varphi \in C^\infty(\mathbb{H}^d)$, where $X_0 = I$. For example, the operator $X_1X_0X_2 = X_1X_2$. If $|\alpha| = 0$, then $X_\alpha\varphi = \varphi$.

We define the Schwartz class $\mathcal{S}(\mathbb{H}^d)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{H}^d)$ on the Heisenberg group as follows:

Definition 1. For any $\varphi \in C^\infty(\mathbb{H}^d)$, we say $\varphi \in \mathcal{S}(\mathbb{H}^d)$ if the function φ satisfies the following condition: For any $N \in \mathbb{Z}_+$, we have $\|\varphi\|_N =$

$$\textstyle\sum_{|\alpha|+l\leq N} \sup_{g\in\mathbb{H}^d}(1 + d_K(g)^2)^l |X_\alpha\varphi(g)| < \infty.$$

It is clear from the definition that the space $\mathcal{S}(\mathbb{H}^d)$ is topologically isomorphic of the space $\mathcal{S}(\mathbb{R}^{2d+1})$. Moreover, it is known that the Schwartz class $\mathcal{S}(\mathbb{H}^d)$ is a Fréchet space in [2].

Definition 2. We denote by $\mathcal{S}'(\mathbb{H}^d)$ the dual space of the space $\mathcal{S}(\mathbb{H}^d)$ and call it the space of the tempered distributions in the Heisenberg group. Thus, $u \in \mathcal{S}'(\mathbb{H}^d)$ if and only if u is a linear functional from $\mathcal{S}(\mathbb{H}^d)$ to $\mathbb{C}$ and satisfies the following condition: There exist $N \in \mathbb{Z}_+$ and a positive constant C such that

$$\left| \langle u, \varphi \rangle \right| \leq C\|\varphi\|_N$$

for any $\varphi \in \mathcal{S}(\mathbb{H}^d)$.

By this definition, we can see that the space $\mathcal{S}'(\mathbb{H}^d)$ is topologically isomorphic of the space $\mathcal{S}'(\mathbb{R}^{2d+1})$.

Let $\check{f}(g) = f(g^{-1})$ for $g \in \mathbb{H}^d$. Then we define the convolution $u * \varphi$ of $u \in \mathcal{S}'(\mathbb{H}^d)$ with $\varphi \in \mathcal{S}(\mathbb{H}^d)$ by $\langle u * \varphi, \psi \rangle = \langle u, \psi * \check{\varphi} \rangle$ for any $\psi \in \mathcal{S}(\mathbb{H}^d)$.

4 The heat kernel method for the space $\mathcal{S}'(\mathbb{H}^d)$

In [5] and [7], we can find the explicit form of the heat kernel (the fundamental solutions) $P_s(g)$ of the heat operator $\partial/\partial s - \Delta_{\mathbb{H}^d}$ on $\mathbb{H}^d$ as follows:

$$P_s(g) = P_s(x, y, t) = \begin{cases} (4\pi s)^{-(d+1)} \displaystyle\int_{-\infty}^{\infty} (2\tau/\sinh 2\tau)^d \times \\ \qquad e^{i\tau t/2s - 2(|x|^2+|y|^2)\tau/(4s\tanh 2\tau)} d\tau, \ s > 0, \\ 0, \ s \leq 0. \end{cases}$$

The following properties of the heat kernel $P_s(g)$ hold:

Proposition 2 ([4])**.** *Let P_s be the heat kernel associated to the sub-Laplacian $\Delta_{\mathbb{H}^d}$. Then the following properties hold:*

1. $P_s(g) \geq 0$,
2. $\displaystyle\int_{\mathbb{H}^d} P_s(g)dg = 1$,
3. $P_s(g) = P_s(g^{-1})$,
4. $(\partial/\partial s - \Delta_{\mathbb{H}^d})\, P_s(g) = 0$,
5. $\lim_{s\to+0} P_s = \delta$ *in* $\mathcal{S}'(\mathbb{H}^d)$,
6. $P_{r^2 s}(rx, ry, r^2 t) = r^{-Q} P_s(x, y, t)$, $r > 0$, $(x, y, t) \in \mathbb{H}^d$.

Moreover the heat kernel $P_s(g)$ has the following estimate:

Proposition 3 ([8])**.** *Let $P_s(g)$ be the heat kernel associated to the sub-Laplacian $\Delta_{\mathbb{H}^d}$. Then for any multi-index $\alpha \in \{0, 1, 2, \cdots, 2d\}^k$ and for any $m \in \mathbb{Z}_+$, there exist positive constants a and $C_{\alpha,m}$ such that*

$$|(\partial/\partial s)^m X_\alpha P_s(g)| \leq C_{\alpha,m} s^{-m-|\alpha|/2-Q/2} e^{-ad_K(g)^2/s}.$$

The following result was introduced in [9]. However there exists no proof. Here we give the proof.

Proposition 4 ([9], [12])**.** *The heat kernel $P_s(g)$ is in the space $\mathcal{S}(\mathbb{H}^d)$ for $s > 0$. Moreover for any $\varphi \in \mathcal{S}(\mathbb{H}^d)$, the following property holds:*

$$U_s \equiv \varphi * P_s \to \varphi \text{ in } \mathcal{S}(\mathbb{H}^d)$$

as s converges to $+0$.

Proof. It is sufficient to show that for any $l \in \mathbb{N}$ and for any multi-index $\alpha \in \{0, 1, 2, \cdots, 2d\}^k$ we have

$$\lim_{s\to+0} \sup_{g\in\mathbb{H}^d} (1 + d_K(g)^2)^l |X_\alpha(\varphi * P_s)(g) - X_\alpha\varphi(g)| = 0.$$

By the definition of the space $\mathcal{S}(\mathbb{H}^d)$, Proposition 2, Proposition 3 and Peetre's inequality,

$$(1+d_K(g)^2) \leq 2(1+d_K(g')^2)(1+(d_K(g)-d_K(g'))^2),$$

we have

$$\begin{aligned}
&|X_\alpha(U_s(g) - \varphi(g))| \\
&= |(\varphi * X_\alpha P_s)(g) - (X_\alpha\varphi)(g)| \\
&= \left| \int_{\mathbb{H}^d} \varphi(g') X_\alpha P_s({g'}^{-1} g) dg' - X_\alpha\varphi(g) \right| \\
&= \left| \int_{\mathbb{H}^d} \varphi(g{g'}^{-1}) X_\alpha P_s(g') dg' - \int_{\mathbb{H}^d} X_\alpha\varphi(g) P_s(g') dg' \right| \\
&\leq \int_{\mathbb{H}^d} |\varphi(g{g'}^{-1})||X_\alpha P_s(g')| dg' + \int_{\mathbb{H}^d} |X_\alpha\varphi(g)||P_s(g')| dg' \\
&\leq \int_{\mathbb{H}^d} C_l (1 + d_K(g{g'}^{-1})^2)^{-l} \times C_\alpha s^{-\frac{|\alpha|}{2}-d-1} e^{\frac{-ad_K(g')^2}{s}} dg' \\
&+ \int_{\mathbb{H}^d} C_{l,\alpha} (1 + d_K(g)^2)^{-l} s^{-d-1} e^{-\frac{ad_K(g')^2}{s}} dg' \\
&\leq C'_{l.\alpha} \int_{\mathbb{H}^d} (1 + (d_K(g) - d_K(g'))^2)^{-l} s^{-\frac{|\alpha|}{2}-d-1} \times \\
&e^{\frac{-ad_K(g')^2}{s}} dg' + C_{l,\alpha}(1 + d_K(g)^2)^{-l} \int_{\mathbb{H}^d} s^{-d-1} e^{-\frac{ad_K(g')^2}{s}} dg' \\
&\leq C'_{l.\alpha} \int_{\mathbb{H}^d} (2^{-1}(1 + d_K(g)^2)(1 + d_K(g')^2)^{-1})^{-l} \times \\
&s^{-\frac{|\alpha|}{2}-d-1} e^{\frac{-ad_K(g')^2}{s}} dg' + C_{l,\alpha}(1 + d_K(g)^2)^{-l} \times \\
&\int_{\mathbb{H}^d} s^{-d-1} e^{-\frac{ad_K(g')^2}{s}} dg' \\
&\leq C''_{l,\alpha} (1 + d_K(g)^2)^{-l} \times \\
&\int_{\mathbb{H}^d} s^{-\frac{|\alpha|}{2}-d-1} (1 + d_K(g')^2)^l e^{-\frac{ad_K(g')^2}{s}} dg'
\end{aligned}$$

for some positive constants $C_l, C_\alpha, C_{l,\alpha}, C'_{l,\alpha}$ and $C''_{l,\alpha}$. Moreover a is the positive constant in

Proposition 3. Hence we obtain the following estimate,

$$
\begin{aligned}
&(1+d_K(g)^2)^l|X_\alpha(\varphi * P_s)(g) - X_\alpha\varphi(g))| \\
&\leq C''_{l,\alpha}\int_{\mathbb{H}^d} s^{-\frac{|\alpha|}{2}-d-1}(1+d_K(g')^2)^l e^{-\frac{ad_K(g')^2}{s}}dg'.
\end{aligned} \tag{4.1}
$$

Now we can see

$$
\lim_{s\to+0} s^{-\frac{|\alpha|}{2}-d-1}e^{-\frac{ad_K(g')^2}{3s}} = 0, \ a.e.\ g' \in \mathbb{H}^d
$$

and

$$
(1+d_K(g')^2)^N e^{-\frac{ad_K(g')^2}{3s}} < M_{a,d,N}
$$

for $0 < s < 1$. Moreover there exist a positive constant $C_{a,d}$ and an integrable function $\prod_{i=1}^{d}\frac{1}{(1+x'^4_i)(1+y'^4_i)(1+t'^2)}$ such that

$$
\begin{aligned}
e^{-\frac{ad_K(g')^2}{3s}} &\leq \frac{C_{a,d}}{(1+d_K(g')^4)^{2^{2d}}} \\
&\leq C_{a,d}\prod_{i=1}^{d}\frac{1}{(1+x'^4_i)(1+y'^4_i)(1+t'^2)},
\end{aligned}
$$

for a.e. $g' \in \mathbb{H}^d$. Therefore by the Lebesque convergence theorem and (4.1), we can see

$$
\lim_{s\to+0}\sup_{g\in\mathbb{H}^d}(1+d_K(g)^2)^l|X_\alpha(U_s(g)-\varphi(g))| = 0.
$$

□

J. Kim and M. W. Wong obtained the following characterization of the space $\mathcal{S}'(\mathbb{H}^d)$. We call this characterization *the heat kernel method for* $\mathcal{S}'(\mathbb{H}^d)$. The reader refers to [9] for the proof of Theorem 1. As a remark, they use Proposition 1 and Proposition 4 in this paper without their proofs in the proof of Theorem 1.

Theorem 1 ([9])**.** *For $u \in \mathcal{S}'(\mathbb{H}^d)$, we put*

$$
U_s(g) = (u * P_s)(g)
$$

for $g \in \mathbb{H}^d$ and $s > 0$. Then the function $U_s(g)$ satisfies the following four conditions:

1. $U_s(g) \in C^\infty(\mathbb{H}^d \times (0,\infty))$,
2. $(\partial/\partial s - \Delta_{\mathbb{H}^d})U_s(g) = 0, \ g \in \mathbb{H}^d$ *and* $s > 0$,
3. *for any* $\varphi \in \mathcal{S}(\mathbb{H}^d)$,

$$
\langle u, \varphi\rangle = \lim_{s\to+0}\int_{\mathbb{H}^d} U_s(g)\varphi(g)dg
$$

and

4. *there exist $\mu, \nu > 0$ and a constant $C > 0$ such that*

$$
|U_s(g)| \leq Cs^{-\mu}(1+\rho(g))^\nu, \ 0 < s < 1,
$$

for $g \in \mathbb{H}^d$.

Conversely every $U_s(g) \in C^\infty(\mathbb{H}^d \times (0,\infty))$ satisfying the conditions 2 and 4 can be expressed in the form

$$
U_s(g) = (u * P_s)(g)
$$

with the unique element $u \in \mathcal{S}'(\mathbb{H}^d)$.

5 The heat kernel method for the space of the tempered distributions supported by a regular closed set on $\mathbb{H}^d$

Here we introduce our recent result in [13]. At first, we give the definition of a regular closed set on $\mathbb{H}^d$.

Definition 3 ([13])**.** Let $A_{\mathbb{H}^d}$ be a closed subset of the set $\mathbb{H}^d = \mathbb{R}^{2d+1}$. If there exist $\kappa > 0$, $\omega > 0$ and $0 < q \leq 1$ such that any g_1 and $g_2 \in A_{\mathbb{H}^d}$ so that $\rho(g_2^{-1}g_1) \leq \kappa$ are linked by a curve in $A_{\mathbb{H}^d}$ whose length l satisfies $l \leq \omega\rho(g_2^{-1}g_1)^q$, then we call $A_{\mathbb{H}^d}$ *a regular* in the Heisenberg group $\mathbb{H}^d$.

We define the space $\mathcal{S}(A_{\mathbb{H}^d})$ as follows:

Definition 4 ([13])**.** Let $A_{\mathbb{H}^d}$ be a regular closed set on $\mathbb{H}^d$. For any $\varphi \in C^\infty(\mathbb{H}^d)$, we say $\varphi \in \mathcal{S}(A_{\mathbb{H}^d})$ if the function φ satisfies the following condition: For any $N \in \mathbb{Z}_+$, we have

$$
\|\varphi\|_{N,A_{\mathbb{H}^d}} = \sup_{g\in A_{\mathbb{H}^d}}(1+\rho(g))^N\sum_{|\alpha|\leq N}|X_\alpha\varphi(g)| < \infty.
$$

The following relationship between the spaces $\mathcal{S}(\mathbb{H}^d)$ and $\mathcal{S}(A_{\mathbb{H}^d})$ holds:

Proposition 5 ([13])**.** *The space $\mathcal{S}(\mathbb{H}^d)$ is dense in the space $\mathcal{S}(A_{\mathbb{H}^d})$.*

Proof. It is enough that the space $\mathcal{D}(\mathbb{H}^d)$ is dense in the space $\mathcal{S}(A_{\mathbb{H}^d})$. We choose $\Theta_j \in \mathcal{D}(\mathbb{H}^d)$ as follows:

$$
\Theta_j(g) = \begin{cases} 1, & \rho(g) \leq j \\ 0, & \rho(g) \geq 2j \end{cases}
$$

for $j = 1, 2, \cdots$. Let f be in $\mathcal{S}(A_{\mathbb{H}^d})$. If we set $\psi_j = f\Theta_j$, the function ψ_j is in $\mathcal{D}(\mathbb{H}^d)$. On the other hand, we have

$X_\alpha\{(1-\Theta_j)f\} = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} X_\beta(1-\Theta_j)X_{\alpha-\beta}f.$

For $\rho(g) \le j$, we can see

$$X_\beta(1-\Theta_j) = 0. \tag{5.1}$$

If the set $A_{\mathbb{H}^d}$ is compact, then by (5.1), we can see that

$$\lim_{j\to\infty} \|f-\psi_j\|_{N,A_{\mathbb{H}^d}} = 0.$$

On the other hand, for unbounded sets $A_{\mathbb{H}^d}$, we obtain the following estimate: For a sufficient large j, we have

$$\begin{aligned}
&\|f-\psi_j\|_{N,A_{\mathbb{H}^d}} = \|(1-\Theta_j)f\|_{N,A_{\mathbb{H}^d}} \\
&\le \sup_{g\in A_{\mathbb{H}^d}\setminus(\{\rho(g)\le j\}\cap A_{\mathbb{H}^d})} (1+\rho(g))^N \times \\
&\qquad \sum_{|\alpha|\le N} |X_\alpha\{(1-\Theta_j)f\}| \\
&\le \sup_{g\in A_{\mathbb{H}^d}\setminus(\{\rho(g)\le j\}\cap A_{\mathbb{H}^d})} (1+\rho(g))^N \times \\
&\qquad \sum_{|\alpha|\le N} |\{X_\alpha(1-\Theta_j)\}f + \cdots \\
&+ X_\zeta(1-\Theta_j)X_\eta f + \cdots + (1-\Theta_j)X_\alpha f|. \qquad (5.2)
\end{aligned}$$

Since $f \in \mathcal{S}(A_{\mathbb{H}^d})$, we can see that

$$(1+\rho(g))^N|X_\eta f| \to 0$$

as $j \to +\infty$. By (5.2), for any $f \in \mathcal{S}(A_{\mathbb{H}^d})$, there exists the sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{H}^d)$ such that

$$\lim_{j\to+\infty} \|f-\psi_j\|_{N,A_{\mathbb{H}^d}} = 0.$$

Therefore we can see that the space $\mathcal{D}(\mathbb{H}^d)$ is dense in the space $\mathcal{S}(A_{\mathbb{H}^d})$. □

Definition 5 ([13])**.** We denote by $\mathcal{S}(A_{\mathbb{H}^d})'$ the dual space of the space $\mathcal{S}(A_{\mathbb{H}^d})$. Thus, $u \in \mathcal{S}(A_{\mathbb{H}^d})'$ if and only if u is a linear functional from $\mathcal{S}(A_{\mathbb{H}^d})$ to $\mathbb{C}$ and satisfies the following condition: There exist $N \in \mathbb{Z}_+$ and a positive constant C such that

$$|\langle u,\varphi\rangle| \le C\|\varphi\|_{N,A_{\mathbb{H}^d}}$$

for any $\varphi \in \mathcal{S}(A_{\mathbb{H}^d})$.

Here we denote by $\mathcal{S}'_{A_{\mathbb{H}^d}}$ the space of the tempered distributions u on $\mathbb{H}^d$ satisfying the following condition: For any $\varphi \in \mathcal{S}(\mathbb{H}^d)$, there exists a constant $C > 0$ such that

$$|\langle u,\varphi\rangle| \le C\|\varphi\|_{N,A_{\mathbb{H}^d}} \tag{5.3}$$

for some $N \in \mathbb{Z}_+$. We call the space $\mathcal{S}'_{A_{\mathbb{H}^d}}$ as the space of the tempered distributions supported by $A_{\mathbb{H}^d}$ in $\mathbb{H}^d$. Then by Proposition 5, (5.3) means that u has continuous on $\mathcal{S}'(\mathbb{H}^d)$ with respect to the relative topology from $\mathcal{S}(A_{\mathbb{H}^d})$. Hence u has a unique linear continuous extension $u_{A_{\mathbb{H}^d}}$ on $\mathcal{S}(A_{\mathbb{H}^d})$. This means that any tempered distributions with supported by $A_{\mathbb{H}^d}$ in $\mathbb{H}^d$ can be identified with an element of $\mathcal{S}(A_{\mathbb{H}^d})'$. Thus, we identify the space $\mathcal{S}'_{A_{\mathbb{H}^d}}$ with the space $\mathcal{S}(A_{\mathbb{H}^d})'$.

We have the following characterization for $\mathcal{S}'(A^d_{\mathbb{H}})$.

Theorem 2 ([13])**.** *Let A be a regular closed set on $\mathbb{H}^d$. For any u in $\mathcal{S}(A_{\mathbb{H}^d})'$, let $U_s(g) = \langle u, P_s(\cdot^{-1}g)\rangle$. Then $U_s(g)$ satisfies the following conditions:*

1. $U_s(g) \in C^\infty(\mathbb{H}^d \times (0,\infty))$,
2. $(\partial/\partial s - \Delta_{\mathbb{H}^d})\, U_s(g) = 0,\ g \in \mathbb{H}^d$ *and* $s > 0$,
3. *for any* $\varphi \in \mathcal{D}(\mathbb{H}^d)$,

   $$\langle u,\varphi\rangle = \lim_{s\to+0} \int_{\mathbb{H}^d} U_s(g)\varphi(g)dg$$

   and
4. *there exist $\mu, \nu > 0$ and a constant $C > 0$ and a such that*

   $$|U_s(g)| \le Cs^{-\mu}(1+\rho(g))^\nu e^{-a\rho(g,A_{\mathbb{H}^d})^2/2s}$$

   for $0 < s < 1$ and $g \in \mathbb{H}^d$, where $\rho(g, A_{\mathbb{H}^d}) = \inf_{g'\in A_{\mathbb{H}^d}} \rho\left(g'^{-1}g\right)$.

Conversely every $U_s(g) \in C^\infty(\mathbb{H}^d \times (0,\infty))$ satisfying the conditions 2 and 4 can be expressed in the form

$$U_s(g) = (u * P_s)(g)$$

with the unique element $u \in \mathcal{S}(A_{\mathbb{H}^d})'$.

6 Conclusion

In this summary, we have introduced the heat kernel method for the tempered distributions on the Heisenberg group making up for their deficiency in [9]. As far the heat kernel method on the Heisenberg group, its investigation has started recently. We will adopt the heat kernel method to the P.D.E. on the Heisenberg group in the future.

Finally, we introduce the Schwartz kernel theorems without the proofs as an application of the heat kernel method as follows:

Theorem 3 ([12]). *Let k be a continuous linear operator from $\mathcal{S}(\mathbb{H}^{d_2})$ to $\mathcal{S}'(\mathbb{H}^{d_1})$. Then there exists T in $\mathcal{S}'(\mathbb{H}^{d_1} \times \mathbb{H}^{d_2})$ such that*

$$\langle k\psi, \varphi \rangle = \langle T, \varphi \otimes \psi \rangle ,$$

where φ is in $\mathcal{S}(\mathbb{H}^{d_1})$ and ψ is in $\mathcal{S}(\mathbb{H}^{d_2})$.

Theorem 4 ([13]). *Let the sets $A_{\mathbb{H}^{d_1}}$ and $A_{\mathbb{H}^{d_2}}$ be regular closed sets on $\mathbb{H}^{d_1}$ and $\mathbb{H}^{d_2}$ respectively and k be a continuous linear operator from $\mathcal{S}(A_{\mathbb{H}^{d_2}})$ to $\mathcal{S}(A_{\mathbb{H}^{d_1}})'$. Then there exists T in $\mathcal{S}(A_{\mathbb{H}^{d_1}} \times A_{\mathbb{H}^{d_2}})'$ such that*

$$\langle k\psi, \varphi \rangle = \langle T, \varphi \otimes \psi \rangle ,$$

where φ is in $\mathcal{S}(A_{\mathbb{H}^{d_1}})$ and ψ is in $\mathcal{S}(A_{\mathbb{H}^{d_2}})$.

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Yasuyuki OKA
General Education, Department of Creative Engineering, National Institute of Technology, Kushiro College,
2-32-1 Otanoshike-Nishi, Kushiro-Shi, Hokkaido 084-0916, Japan.
e-mail address: oka@kushiro-ct.ac.jp