Negative dimensional integral technique and its extension

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Abstract

We introduce Negative dimensional integral technique devised by I.G.Halliday and R.M.Ricotta. And we extend the idea to all dimensions using complex variable integral and negative power differentiation. Furthermore we propose a new formula to calculate Feynman integrals. We apply it to the calculations of bubble Feynman integral.

Key Words : Feynman integrals, Negative dimension, Hyper-geometric function, Fractional calculus

1 Introduction

Until now, many kinds of ways to calculate Feynman propagator integrals were discovered. Dimensional regularization method, founded by G.'tHooft and M.Veltman, was the most successful one in Quantum field theory.[2] In 1987, I.G.Halliday and R.M.Ricotta conceived an idea known as negative dimensional integral method.[3] It was advanced by C.Anastasiou and others, A.T.Suzuki, A.G.M.Schmit as well as other researchers.[1][4][5] When we calculated Feynman integral propagators, we proposed a new parameter transformation and a new regularization, the so-called Hypersurface regularization.[6][7][9] In this paper we extend the idea of negative dimension to all dimensions. Using our idea and our calculation method of Feynman propagator integrals, we calculate bubble one loop Feynman integral. We show that the results perfectly agree with the results of the standard calculation. In Sec. 2 we introduce negative dimensional integral method and explain our idea as the extension of this method. In Sec. 3 we apply our idea to bubble Feynman diagram integral calculation with consideration of C.Anastasiou, E.W.N.Glover and C.Oleari.[1] In Sec. 4 we discuss the advantages and the doubtful points concerning our new idea and the new formula, and explain prospects for the future as concluding remarks.

2 Negative dimensional integral technique and its extension

First, we review negative dimensional integral technique, discovered by Halliday and Ricotta, and try to extend it. Furthermore we will make a new formula to calculate Feynman propagator integrals and verify that its formula is exactly true.

Let's consider the following integral,

$$\int d^d p \frac{(p^2)^\alpha}{(p^2+M^2)^\beta} = \pi^{\frac{d}{2}} (M^2)^{\frac{d}{2}+\alpha-\beta} \frac{\Gamma(\alpha+d/2)\Gamma(\beta-\alpha-d/2)}{\Gamma(d/2)\Gamma(\beta)}. \tag{1}$$

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We can prove this formula by utilizing the d-dimensional hyper-spherical coordinate integral as follows,

$$\int d^dp \frac{(p^2)^\alpha}{(p^2+M^2)^\beta} = \int_\Omega d\Omega_{d-1} \int_0^\infty \frac{(p^2)^\alpha}{(p^2+M^2)^\beta} (p^2)^{\frac{d-1}{2}} \frac{1}{2} (p^2)^{-\frac{1}{2}} d(p^2)$$
$$= \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty \frac{(M^2)^\alpha \left(\frac{p^2}{M^2}\right)^\alpha}{(M^2)^\beta \left(1+\frac{p^2}{M^2}\right)^\beta} \left(\frac{p^2}{M^2}\right)^{\frac{d}{2}-1} (M^2)^{\frac{d}{2}-1} (M^2) d\left(\frac{p^2}{M^2}\right) = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} (M^2)^{\alpha-\beta+\frac{d}{2}} \int_0^\infty \frac{x^{\alpha+\frac{d}{2}-1}}{(1+x)^\beta} dx$$
$$= \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} (M^2)^{\alpha-\beta+\frac{d}{2}} B(\alpha+d/2, \beta-\alpha-d/2) = \pi^{d/2} (M^2)^{\alpha-\beta+\frac{d}{2}} \frac{\Gamma(\alpha+d/2)\Gamma(\beta-\alpha-d/2)}{\Gamma(d/2)\Gamma(\beta)}. \tag{2}$$

Now we consider the integral $\int (p^2)^\alpha d^dp$ in the case of $\beta = 0$. The result becomes 0 because $\Gamma(0)$ is infinity. In order to avoid this difficulty we have to take $\alpha + \frac{d}{2} = 0$. Accordingly, because α is positive, the dimension d must be negative. This is an idea of negative dimension, proposed by Halliday and Ricotta.[3] This idea means an analytic continuation to negative dimension as Feynman propagator integrals are analytic concerning the dimension.
We show the concrete calculation as follows,

$$\int (p^2)^\alpha d^dp = \lim_{\beta\to 0} \int \frac{(p^2)^\alpha}{(p^2+M^2)^\beta} d^dp$$
$$= \lim_{\beta\to 0} \pi^{\frac{d}{2}} (M^2)^{\alpha-\beta+\frac{d}{2}} \frac{\Gamma(\alpha+d/2)\Gamma(\beta-\alpha-d/2)}{\Gamma(d/2)\Gamma(\beta)} = (-1)^\alpha \pi^{\frac{d}{2}} \Gamma(\alpha+1) \delta_{\alpha+\frac{d}{2},0}, \tag{3}$$

where we used the formula,[8]

$$\frac{\Gamma(\alpha+d/2)}{\Gamma(d/2)} = \left(\frac{d}{2}\right)_\alpha = (-\alpha)_\alpha \tag{4}$$
$$= (-\alpha)(-\alpha+1)\cdots\cdots(-\alpha+\alpha-1) = (-1)^\alpha 1 \times 2 \times \cdots\cdots \times (\alpha-1) \times \alpha = (-1)^\alpha \Gamma(\alpha+1).$$

The negative dimensional technique became a powerful tool. When we have an integral, such as the above type Eq. (3), it is sufficient to replace the integral by $(-1)^\alpha \pi^{\frac{d}{2}} \Gamma(\alpha+1) \delta_{\alpha+\frac{d}{2},0}$.

Next, we examine another aspect of negative dimensional technique. This time, we introduce a complex parameter variable z and parametrize the above integral as the contour integral along a small circle C around the origin $z = 0$ on the complex plane. Because $\exp(-p^2 z)$ does not have any singularities and is holomorphic at $z = 0$, the following equation holds

$$\int (p^2)^\alpha d^dp = \int d^dp \frac{(-1)^\alpha (\alpha)!}{2\pi i} \oint_C \frac{dz}{z^{\alpha+1}} \exp(-p^2 z) = \frac{\alpha!(-1)^\alpha}{2\pi i} \oint_C \frac{1}{z^{\alpha+1}} \left(\frac{\pi}{z}\right)^{\frac{d}{2}} dz$$
$$= \frac{\alpha!(-1)^\alpha}{2\pi i} \oint_C \frac{\pi^{\frac{d}{2}}}{z^{\alpha+1+\frac{d}{2}}} dz = (-1)^\alpha \pi^{\frac{d}{2}} \Gamma(\alpha+1) \delta_{\alpha+\frac{d}{2},0}, \tag{5}$$

applying Cauchy's integral theorem concerning the function with $\alpha+1$ order pole at $z = 0$. We can understand that negative dimensional integral method is consistent with the concept that a single pole only is effective on the integral calculation of the complex variable parameter z.
Subsequently we have to extend negative dimensional method to all dimensional one by considering that the following equation

$$\int \frac{d^dp}{(p^2)^\alpha} = \int d^dp \frac{(-1)^\alpha (-\alpha)!}{2\pi i} \oint_C \frac{dz}{z^{1-\alpha}} \exp(-p^2 z) = \frac{(-\alpha!)(-1)^\alpha}{2\pi i} \oint_C \frac{1}{z^{1-\alpha}} \left(\frac{\pi}{z}\right)^{\frac{d}{2}} dz$$
$$= (-1)^\alpha \pi^{\frac{d}{2}} \Gamma(1-\alpha) \delta_{-\alpha+\frac{d}{2},0} \tag{6}$$

holds.
In this extension we adopted the concept that $-n$ times differentiation means n times integration.[10]
Henceforth we postulate the following items:

(a) we can exchange the order of integration of the variable p and the parameter complex variable z.

(b) The single pole only is effective for the contour complex integral after expanding the integrand to Taylor expansion or a multinomial one.

(c) We interpret that negative n times differentiation means n times integration.

(d) We can do analytic continuation of the Feynman propagator integral to all dimensions from positive dimension until negative dimension.

(e) The sign and the domain of the indexes in the expansions can be decided by the mathematical constraints and the physical constraints.

We try to calculate the following integral, utilizing our new formula, in order to examine whether our new formula holds or not. We have the following integral from Eq. (7) in Appendix C-2 of Ref. [11],

$$I = \frac{1}{i(2\pi)^d}\int d^dk \frac{1}{(m^2+2k{\cdot}p-k^2)^\alpha} = \frac{\Gamma(\alpha-d/2)}{(4\pi)^{\frac{d}{2}}\Gamma(\alpha)}\frac{1}{(m^2+p^2)^{\alpha-\frac{d}{2}}}. \tag{7}$$

We diagonalize the momentum k and shift it as follows $\hat{k}=k-p$. Then the process of the calculation can be shown as

$$\begin{aligned} I &= \int d^dk \frac{1}{(m^2+2kp-k^2)^\alpha} = \int d^d\hat{k}\frac{1}{(-\hat{k}^2+p^2+m^2)^\alpha} \\ &= \int d^d\hat{k}\frac{(-\alpha)!}{2\pi i}\oint_C \frac{dz}{z^{1-\alpha}}\exp{(-\hat{k}^2+p^2+m^2)z} = \frac{(-\alpha)!}{2\pi i}\oint_C \frac{dz}{z^{1-\alpha}}\left(\frac{\pi}{z}\right)^{\frac{d}{2}}\exp{(p^2+m^2)z}. \end{aligned} \tag{8}$$

Finally we can obtain the result as follows,

$$\begin{aligned} I &= (-\alpha)!\pi^{\frac{d}{2}}\sum_{n=0}^{\infty}\frac{(p^2+m^2)^n}{n!}\frac{1}{2\pi i}\oint_C\frac{dz}{z^{1-\alpha+\frac{d}{2}-n}} = (-\alpha)!\pi^{\frac{d}{2}}\sum_{n=0}^{\infty}\frac{(p^2+m^2)^n}{n!}\delta_{n+\alpha-\frac{d}{2},0} \\ &= \pi^{\frac{d}{2}}\frac{\Gamma(1-\alpha)}{\Gamma(1+d/2-\alpha)}(p^2+m^2)^{\frac{d}{2}-\alpha} = \pi^{\frac{d}{2}}(-1)^{\frac{d}{2}}\frac{\Gamma(\alpha-d/2)}{\Gamma(\alpha)}\frac{1}{(p^2+m^2)^{\alpha-\frac{d}{2}}} \end{aligned} \tag{9}$$

This final result is consistent with one of the usual calculations except for a factor $(-1)^{\frac{d}{2}}$. In the calculation of the final term, we utilized the formula concerning Γ-function;

$$\frac{\Gamma(1-\alpha)}{\Gamma(1+d/2-\alpha)} = \frac{\Gamma(\alpha-d/2)(-1)^{\alpha-\frac{d}{2}}}{\Gamma(\alpha)(-1)^\alpha} \tag{10}$$

3 The application to massive bubble Feynman diagram calculation

Now we consider application of our idea and the new formula to the calculation of massive bubble Feynman diagram as indicated in Fig.1. This one loop integral in d-dimensional Minkowski space is given

$$I_d^{\{2\}} = \int\frac{d^dk}{i\pi^{\frac{d}{2}}}\frac{1}{[k^2-M_1^2][(k+k_1)^2-M_2^2]}, \tag{11}$$

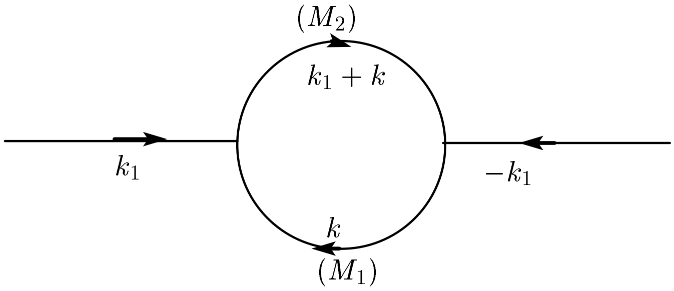

Fig.1

where the external momenta are k_1,$k_2 = -k_1$, all incoming, and the masses of propagators are M_1, M_2. We omit the infinitesimal quantity $i\epsilon$ for short. We install the power indexes ν_1,ν_2 in the denominator to generalize it as possible as we can,

$$I_d^{\{2\}} = \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{[k^2 - M_1^2]^{\nu_1}[(k+k_1)^2 - M_2^2]^{\nu_2}}. \tag{12}$$

Of course we can take $\nu_1 = 1$, $\nu_2 = 1$ if we need, after we have calculated the integral. We transform the momentum k in Minkowski space to the momentum $\tilde{k}$ in Euclidean space and apply our formula Eq. (6) to Eq. (12).
That is,

$$I_d^{\{2\}} = \int \frac{d^d \tilde{k}}{\pi^{\frac{d}{2}}} \frac{(-\nu_1)!(-\nu_2)!}{(2\pi i)^2} \oint_C \frac{dz_1 dz_2}{z_1^{1-\nu_1} z_2^{1-\nu_2}} \exp\left[-(\tilde{k}^2 + M_1^2)z_1 - ((\tilde{k}+\tilde{k_1})^2 + M_2^2)z_2\right]. \tag{13}$$

We diagonalize the momentum $\tilde{k}$ and sift the momentum from $\tilde{k}$ to $\hat{k} = \tilde{k} + \frac{z_2}{z_1 z_2}\tilde{k}_1$ and exchange the integral order of z_i and $\hat{k}$. After that we calculate the integral of $\hat{k}$ first.
Namely we have

$$\begin{aligned} I_d^{\{2\}} &= \frac{(-\nu_1)!(-\nu_2)!}{(2\pi i)^2} \oint_C \frac{dz_1 dz_2}{z_1^{1-\nu_1} z_2^{1-\nu_2}} \int \frac{d^d \hat{k}}{\pi^{\frac{d}{2}}} \exp\left[-(z_1+z_2)\hat{k}^2\right] \\ &\quad \times \exp\left[-\frac{z_1 z_2}{z_1+z_2}\tilde{k_1}^2 - (M_1^2 z_1 + M_2^2 z_2)\right] \\ &= \frac{(-1)^{\nu_1}(-1)^{\nu_2}}{\pi^{d/2}}(-\nu_1)!(-\nu_2)! \frac{1}{(2\pi i)^2} \oint \frac{dz_1 dz_2}{z^{1-\nu_1} z^{1-\nu_2}} \left(\frac{\pi}{z_1+z_2}\right)^{\frac{d}{2}} \\ &\quad \times \exp\left[\frac{z_1 z_2}{z_1+z_2}k_1^2 - M_1^2 z_1 - M_2^2 z_2\right]. \end{aligned} \tag{14}$$

Next we carry out Taylor expansion and multinomial expansions in the integrand as follows,

$$\begin{aligned} I_d^{\{2\}} &= (-1)^{\nu_1}(-1)^{\nu_2}(-\nu_1)!(-\nu_2)! \frac{1}{(2\pi i)^2} \oint_C \frac{dz_1 dz_2}{z^{1-\nu_1} z^{1-\nu_2}} (z_1+z_2)^{-\frac{d}{2}} \\ &\quad \times \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{z_1 z_2}{z_1+z_2}k_1^2 - M_1^2 z_1 - M_2^2 z_2\right]^n \\ &= (-1)^{\nu_1}(-1)^{\nu_2}(-\nu_1)!(-\nu_2)! \frac{1}{(2\pi i)^2} \oint_C \frac{dz_1 dz_2}{z^{1-\nu_1} z^{1-\nu_2}} (z_1+z_2)^{-\frac{d}{2}} \\ &\quad \times \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\{n_1,n_2,n_3\}} \frac{n!}{n_1! n_2! n_3!} \Big(\frac{z_1 z_2}{z_1+z_2}k_1^2\Big)^{n_1} \Big(-M_1^2 z_1\Big)^{n_2} \Big(-M_2^2 z_2\Big)^{n_3} \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{\nu_1}(-1)^{\nu_2}(-\nu_1)!(-\nu_2)!\sum_{n=0}^{\infty}\sum_{\{n_1,n_2,n_3\}}\frac{(k_1^2)^{n_1}(-M_1^2)^{n_2}(-M_2^2)^{n_3}}{n_1!n_2!n_3!} \\
&\quad\times\frac{1}{(2\pi i)^2}\oint_C\frac{dz_1dz_2}{z_1^{1-\nu_1-n_1-n_2}z_2^{1-\nu_2-n_1-n_3}}\Big(z_1+z_2\Big)^{-n_1-\frac{d}{2}} \\
&= (-1)^{\nu_1}(-1)^{\nu_2}(-\nu_1)!(-\nu_2)!\sum_{n=0}^{\infty}\sum_{\{n_1,n_2,n_3,q_1,q_2\}} \\
&\quad\frac{(k_1^2)^{n_1}(-M_1^2)^{n_2}(-M_2^2)^{n_3}}{n_1!n_2!n_3!q_1!q_2!}\frac{(-n_1-d/2)!}{(2\pi i)^2}\oint_C\frac{dz_1dz_2}{z_1^{1-\nu_1-n_1-n_2-q_1}z_2^{1-\nu_2-n_1-n_3-q_2}}. \qquad (15)
\end{aligned}
$$

Finally we perform the complex variable z_1, z_2 integrations along the contours C_1, C_2 around the small circles at the centers of the origins $z_1 = 0, z_2 = 0$, respectively. We can get the following result because only the single poles are effective, from Cauchy integral theorem.
That is,

$$
\begin{aligned}
I_d^{\{n,q\}} &= (-1)^{\nu_1}(-1)^{\nu_2}(-\nu_1!)(-\nu_2)!\sum_{\{n_1,n_2,n_3,q_1,q_2\}} \\
&\quad\frac{(k_1^2)^{n_1}(-M_1^2)^{n_3}(-M_2^2)^{n_3}(-n_1-d/2)!}{n_1!n_2!n_3!q_1!q_2!}\delta_{\nu_1+n_1+n_2+q_1,0}\cdot\delta_{\nu_2+n_1+n_3+q_2,0}. \qquad (16)
\end{aligned}
$$

Hence we can obtain the constraint conditions as follows,

$$
n_1+n_2+n_3=n,\ \ n_1+n_2+q_1=-\nu_1,\ \ n_1+n_3+q_2=-\nu_2,\ \ q_1+q_2+n_1=-\frac{d}{2}. \qquad (17)
$$

Because we have $n = n_1+n_2+n_3 = \frac{d}{2}-\nu_1-\nu_2 =$ constant, the undetermined indexes are the five ones of n_1, n_2, n_3, q_1, q_2 . Therefore we can solve the four simultaneous equations concerning any two indexes out of the five ones. The combinations are $(n_1, n_2), (n_2, n_3), (n_1, n_3), (n_1, q_1), (n_1, q_2), (n_2, q_1), (n_3, q_2), (q_1, q_2)$. The combinations $(n_2, q_1), (n_3, q_2)$ cannot be determined because they are not linear independent of each other.
We enumerate the concrete procedures and results of the computation as (i)(ii)(iii)(iv)(v)(vi)(vii)(viii).

(i) the case $\{n_1, n_2\}$

Solving Eq. (17) on n_1, n_2, we can obtain the following relations;

$$
n_3=\frac{d}{2}-\nu_1-\nu_2-n_1-n_2,\quad q_1=-\nu_1-n_1-n_2,\quad q_2=\nu_1-\frac{d}{2}+n_2. \qquad (18)
$$

Substituting these relations to Eq. (16) yields

$$
\begin{aligned}
I_d^{\{n_1,n_2\}} &= (-1)^{\nu_1}(-1)^{\nu_2}(-\nu_1)!(-\nu_2)! \\
&\quad\times\sum_{\{n_1,n_2\}}\frac{(k_1^2)^{n_1}(-M_1^2)^{n_2}(-M_2^2)^{d/2-\nu_1-\nu_2-n_1-n_2}}{n_1!n_2!(d/2-\nu_1-\nu_2-n_1-n_2)!(-\nu_1-n_1-n_2)!}\times\frac{(-n_1-d/2)!}{(\nu_1+n_2-d/2)!} \\
&= (-1)^{\nu_1}(-1)^{\nu_2}\Gamma(1-\nu_1)\Gamma(1-\nu_2)(-M_2^2)^{\frac{d}{2}-\nu_1-\nu_2} \\
&\quad\times\sum_{\{n_1,n_2\}}\frac{\left(\frac{-k_1^2}{M_2^2}\right)^{n_1}\left(\frac{M_1^2}{M_2^2}\right)^{n_2}}{n_1!n_2!\Gamma(1+d/2-\nu_1-\nu_2)(1+d/2-\nu_1-\nu_2)_{-(n_1+n_2)}} \\
&\quad\times\frac{\Gamma(1-d/2)(1-d/2)_{-(n_1)}}{\Gamma(1-\nu_1)(1-\nu_1)_{-(n_1+n_2)}\Gamma(1+\nu_1-d/2)(1+\nu_1-d/2)_{n_2}}
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{\nu_1}(-1)^{\nu_2}(-M_2^2)^{d/2-\nu_1-\nu_2}\frac{\Gamma(1-\nu_2)\Gamma(1-d/2)}{\Gamma(1+d/2-\nu_1-\nu_2)\Gamma(1+\nu_1-d/2)} \\
&\quad\times\sum_{\{n_1,n_2\}}\frac{(\nu_1+\nu_2-d/2)_{n_1+n_2}(\nu_1)_{n_1+n_2}(-1)^{n_1}}{(d/2)_{n_1}(1+\nu_1-d/2)_{n_2}}\Big(-\frac{k_1^2}{M_2^2}\Big)^{n_1}\Big(\frac{M_1^2}{M_2^2}\Big)^{n_2} \\
&= (-1)^{\nu_1}(-1)^{\nu_2}(-1)^{d/2}(-M_2^2)^{d/2-\nu_1-\nu_2}\frac{\Gamma(d/2-\nu_1)\Gamma(\nu_1+\nu_2-d/2)}{\Gamma(\nu_2)\Gamma(d/2)} \\
&\quad\times F_4\Big(\nu_1,\nu_1+\nu_2-\frac{d}{2};\frac{d}{2},1+\nu_1-\frac{d}{2};\frac{k_1^2}{M_2^2},\frac{M_1^2}{M_2^2}\Big), \qquad (19)
\end{aligned}
$$

where we made use of the formula $(a+n)!=\Gamma(1+a)(1+a)_n$ and $(a)_{-n}=\frac{(-1)^n}{(1-a)_n}$ and $F_4(a,a';b,c;x,y)$ is a hyper-geometric function with two variables, known as Appell function, i.e.$F_4(a,b;c,c';x,y)=\sum_{m,n}\frac{(a)_{m+n}(b)_{m+n}}{m!n!(c)_m(c')_n}x^m y^n \quad (|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}<1)$.[8] In the last line, the relation $\frac{\Gamma(n)\Gamma(1-n)}{\Gamma(m)\Gamma(1-m)}=\frac{(-1)^m}{(-1)^n}$ has been used. In the same way, we can calculate the rest seven cases. The results are enumerated below.

(ii) the case $\{n_2,n_3\}$
The result:

$$
\begin{aligned}
I_d^{\{n_2,n_3\}} &= (-1)^{\nu_1}(-1)^{\nu_2}\frac{\Gamma(1-\nu_1)\Gamma(1-\nu_2)\Gamma(1-d+\nu_1+\nu_2)}{\Gamma(1+d/2-\nu_1-\nu_2)\Gamma(1+\nu_1-d/2)\Gamma(1+\nu_2-d/2)} \\
&\quad\times(k_1^2)^{\frac{d}{2}-\nu_1-\nu_2}\sum_{\{n_2,n_3\}}\frac{(\nu_1+\nu_2-d/2)_{n_2+n_3}(1-d+\nu_1+\nu_2)_{n_2+n_3}}{n_2!n_3!(1+\nu_1-d/2)_{n_2}(1+\nu_2-d/2)_{n_3}}\left(\frac{M_1^2}{k_1^2}\right)^{n_2}\left(\frac{M_2^2}{k_1^2}\right)^{n_3} \\
&= (-1)^{\nu_1}(-1)^{\nu_2}(-1)^{d/2}\frac{\Gamma(\nu_1+\nu_2-d/2)\Gamma(d/2-\nu_1)\Gamma(d/2-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(d-\nu_1-\nu_2)} \\
&\quad\times(k_1^2)^{\frac{d}{2}-\nu_1-\nu_2}F_4\Big(\nu_1+\nu_2-\frac{d}{2},1-d+\nu_1+\nu_2;1+\nu_1-\frac{d}{2},1+\nu_2-\frac{d}{2};\frac{M_1^2}{k_1^2},\frac{M_2^2}{k_1^2}\Big)
\end{aligned}
$$
(20)

(iii) the case $\{n_1,n_3\}$
The result:

$$
\begin{aligned}
I_d^{\{n_1,n_3\}} &= (-1)^{\nu_1}(-1)^{\nu_2}(-M_1^2)^{\frac{d}{2}-\nu_1-\nu_2}\frac{\Gamma(1-\nu_1)\Gamma(1-d/2)}{\Gamma(1+d/2-\nu_1-\nu_2)\Gamma(1+\nu_2-d/2)} \\
&\quad\times\sum_{\{n_1,n_3\}}\frac{(-1)^{n_1}(\nu_1+\nu_2-d/2)_{n_1+n_3}(\nu_2)_{n_1+n_3}}{(d/2)_{n_1}(1+\nu_2-d/2)_{n_3}}\left(-\frac{k_1^2}{M_1^2}\right)^{n_1}\left(\frac{M_2^2}{M_1^2}\right)^{n_3} \\
&= (-1)^{\nu_1}(-1)^{\nu_2}(-1)^{d/2}\frac{\Gamma(\nu_1+\nu_2-d/2)\Gamma(d/2-\nu_2)}{\Gamma(\nu_1)\Gamma(d/2)} \\
&\quad\times(-M_1^2)^{\frac{d}{2}-\nu_1-\nu_2}F_4\Big(\nu_2,\nu_1+\nu_2-\frac{d}{2};\frac{d}{2},1+\nu_2-\frac{d}{2};\frac{k_1^2}{M_1^2},\frac{M_2^2}{M_1^2}\Big). \qquad (21)
\end{aligned}
$$

(iv) the case $\{n_1,q_1\}$
The result:

$$
\begin{aligned}
I_d^{\{n_1,q_1\}} &= (-1)^{\nu_1}(-1)^{\nu_2}\frac{\Gamma(1-\nu_1)\Gamma(1-\nu_2)\Gamma(1-d/2)}{\Gamma(1-\nu_1)\Gamma(1+d/2-\nu_2)\Gamma(1-d/2)}(-M_1^2)^{-\nu_1}(-M_2^2)^{\frac{d}{2}-\nu_2} \\
&\quad\times\sum_{\{n_1,q_1\}}\frac{(-1)^{n_1}(\nu_1)_{n_1+q_1}(d/2)_{n_1+q_1}}{n_1!q_1!(1+d/2-\nu_2)_{q_1}(d/2)_{n_1}}\left(-\frac{k_1^2}{M_1^2}\right)^{n_1}\left(\frac{M_2^2}{M_1^2}\right)^{q_1}
\end{aligned}
$$

$$= (-1)^{\nu_1}(-1)^{\nu_2}(-1)^{d/2}\frac{\Gamma(\nu_2-d/2)}{\Gamma(\nu_2)}(-M_1^2)^{-\nu_1}(-M_2^2)^{\frac{d}{2}-\nu_2}$$

$$\times F_4\Big(\nu_1,\frac{d}{2};\frac{d}{2},1+\frac{d}{2}-\nu_2;\frac{k_1^2}{M_1^2},\frac{M_2^2}{M_1^2}\Big). \quad (22)$$

(v) the case $\{n_1, q_2\}$
The result:

$$I_d^{\{n_1,q_2\}} = (-1)^{\nu_1}(-1)^{\nu_2}\frac{\Gamma(1-\nu_1)\Gamma(1-\nu_2)\Gamma(1-d/2)}{\Gamma(1+d/2-\nu_1)\Gamma(1-\nu_2)\Gamma(1-d/2)}(-M_1^2)^{\frac{d}{2}-\nu_1}(-M_2^2)^{-\nu_2}$$

$$\times\sum_{\{n_1,q_2\}}\frac{(-1)^{n_1}(\nu_2)_{n_1+q_2}(d/2)_{n_1+q_2}}{n_1!q_2!(d/2)_{n_1}(1+d/2-\nu_1)_{q_2}}\left(-\frac{k_1^2}{M_2^2}\right)^{n_1}\left(\frac{M_1^2}{M_2^2}\right)^{q_2}$$

$$= (-1)^{\nu_1}(-1)^{\nu_2}(-1)^{d/2}\frac{\Gamma(\nu_1-d/2)}{\Gamma(\nu_1)}(-M_1^2)^{\frac{d}{2}-\nu_1}(-M_2^2)^{-\nu_2}$$

$$\times F_4\Big(\nu_2,\frac{d}{2};\frac{d}{2},1+\frac{d}{2}-\nu_1;\frac{k_1^2}{M_2^2},\frac{M_1^2}{M_2^2}\Big). \quad (23)$$

(vi) the case $\{n_2, q_1\}$
The result:

$$I_d^{\{n_2,q_1\}} = (-1)^{\nu_1}(-1)^{\nu_2}(k_1^2)^{-\nu_1}(-M_2^2)^{-\nu_2+\frac{d}{2}}\frac{\Gamma(1-\nu_1)\Gamma(1-\nu_2)\Gamma(1+\nu_1-d/2)}{\Gamma(1-\nu_1)\Gamma(1+d/2-\nu_2)\Gamma(1+\nu_1-d/2)}$$

$$\times\sum_{\{n_2,q_1\}}\frac{(\nu_1)_{n_2+q_1}(1+\nu_1-d/2)_{n_2+q_1}}{n_2!q_1!(-1)^{n_2+q_1}(1+\nu_1-d/2)_{n_2}(1-\nu_2+d/2)_{q_1}}\left(-\frac{M_2^2}{k_1^2}\right)^{n_2}\left(-\frac{M_1^2}{k_1^2}\right)^{q_1}$$

$$= (-1)^{\nu_1}(-1)^{\nu_2}(-1)^{d/2}\frac{\Gamma(\nu_2-d/2)}{\Gamma(\nu_2)}(-M_2^2)^{\frac{d}{2}-\nu_2}$$

$$\times F_4\Big(\nu_1,1+\nu_1-\frac{d}{2};1+\nu_1-\frac{d}{2},1-\nu_2+\frac{d}{2};\frac{M_1^2}{k_1^2},\frac{M_2^2}{k_1^2}\Big) \quad (24)$$

(vii) the case $\{n_3, q_2\}$
The result:

$$I_d^{\{n_3,q_2\}} = (-1)^{\nu_1}(-1)^{\nu_2}(k_1^2)^{-\nu_2}(-M_1^2)^{\frac{d}{2}-\nu_1}\frac{\Gamma(1-\nu_1)\Gamma(1-\nu_2)\Gamma(1-d/2+\nu_2)}{\Gamma(1-\nu_2)\Gamma(1+d/2-\nu_1)\Gamma(1+\nu_2-d/2)}$$

$$\times\sum_{\{n_3,q_2\}}\frac{(\nu_2)_{n_3+q_2}(1+\nu_2-d/2)_{n_3+q_2}}{n_3!q_2!(-1)^{n_3+q_2}(1+\nu_2-d/2)_{n_3}(1-\nu_1+d/2)_{q_2}}\left(-\frac{M_2^2}{k_1^2}\right)^{n_3}\left(-\frac{M_1^2}{k_1^2}\right)^{q_2}$$

$$= (-1)^{\nu_1}(-1)^{\nu_2}(-1)^{d/2}\frac{\Gamma(\nu_1-d/2)}{\Gamma(\nu_1)}(k_1^2)^{-\nu_2}(-M_1^2)^{\frac{d}{2}-\nu_1}$$

$$\times F_4\Big(\nu_2,1+\nu_2-\frac{d}{2};1+\nu_2-\frac{d}{2},1-\nu_1+\frac{d}{2};\frac{M_2^2}{k_1^2},\frac{M_1^2}{k_1^2}\Big). \quad (25)$$

(viii) the case $\{q_1, q_2\}$
The result:

$$I_d^{\{q_1,q_2\}} = (-1)^{\nu_1}(-1)^{\nu_2}\frac{\Gamma(1-\nu_1)\Gamma(1-\nu_2)}{\Gamma(1-d/2)\Gamma(1+d/2-\nu_1)\Gamma(1+d/2-\nu_2)}(k_1^2)^{-\frac{d}{2}}(-M_1^2)^{\frac{d}{2}-\nu_1}(-M_2^2)^{\frac{d}{2}-\nu_2}$$

$$\times\sum_{\{q_1,q_2\}}\frac{(d/2)_{q_1+q_2}(1)_{q_1+q_2}}{q_1!q_2!(-1)^{q_1+q_2}(1+d/2-\nu_2)_{q_1}(1+d/2-\nu_1)_{q_2}}\left(-\frac{M_2^2}{k_1^2}\right)^{q_1}\left(-\frac{M_1^2}{k_1^2}\right)^{q_2}$$

Table-1

	positive	negative		positive	negative
(a)	n_1, n_2	n_3	(c)	n_1, n_3	q_2
	n_2, n_3	n_1		n_1, q_2	n_3
	n_1, n_3	n_2		n_3, q_2	n_1
(b)	n_1, n_2	q_1	(d)	n_1, q_1	q_2
	n_1, q_1	n_2		n_1, q_2	q_1
	n_2, q_1	n_1		q_1, q_2	n_1

$$
\begin{aligned}
&= (-1)^{\nu_1}(-1)^{\nu_2}(-1)^{d/2}\frac{\Gamma(d/2)\Gamma(\nu_1-d/2)\Gamma(\nu_2-d/2)}{\Gamma(\nu_1)\Gamma(\nu_2)}(k_1^2)^{-\frac{d}{2}}(-M_1^2)^{\frac{d}{2}-\nu_1}(-M_2^2)^{\frac{d}{2}-\nu_2}\\
&\times F_4\Big(1,\frac{d}{2};1+\frac{d}{2}-\nu_2,1+\frac{d}{2}-\nu_1;\frac{M_2^2}{k_1^2},\frac{M_1^2}{k_1^2}\Big). \qquad (26)
\end{aligned}
$$

Furthermore, we can classify the results of the calculation to three groups by considering that Appell function $F_4(a,b;c,c';x,y)$ is convergent only when $\sqrt{|x|}+\sqrt{|y|}<1$ holds.
The classification of $I_d^{\{2\}}$ becomes as follows,

$$
(a)\ \ I_d^{\{2\}}(\nu_1,\nu_2;k_1^2,M_1^2,M_2^2) = I_d^{\{n_2,n_3\}} + I_d^{\{n_2,q_1\}} + I_d^{\{n_3,q_2\}} + I_d^{\{q_1,q_2\}} \qquad (27)
$$

$$
\text{when}\sqrt{M_1^2}+\sqrt{M_2^2}<\sqrt{k_1^2}
$$

$$
(b)\ \ I_d^{\{2\}} = I_d^{\{2\}}(\nu_1,\nu_2;k_1^2,M_1^2,M_2^2) = I_d^{\{n_1,n_3\}} + I_d^{\{n_1,q_1\}} \qquad (28)
$$

$$
\text{when } \sqrt{k_1^2}+\sqrt{M_2^2}<\sqrt{M_1^2}
$$

$$
(c)\ \ I_d^{\{2\}} = I_d^{\{2\}}(\nu_1,\nu_2;k_1^2,M_1^2,M_2^2) = I_d^{\{n_1,n_2\}} + I_d^{\{n_1,q_2\}} \qquad (29)
$$

when $\sqrt{k_1^2}+\sqrt{M_1^2}<\sqrt{M_2^2}$. Next we introduce an amazing idea of the classification by C.Anastasiou, et al.[1] It is supposed that the indexes $n_i, q_j (i=1,2,3, j=1,2)$ are very large and so $\nu_i (i=1,2)$, $\frac{d}{2}$ can be negligible. In this case the constraint conditions become as follows,

$$
n_1+n_2+n_3=0\ \ (a),\ \ n_1+n_2+q_1=0\ \ (b),\ \ n_1+n_3+q_2=0\ \ (c),\ \ n_1+q_1+q_2=0\ \ (d) \qquad (30)
$$

From Equation (a) when (n_1,n_2),(n_2,n_3),(n_1,n_3) respectively are positive, n_3,n_1 and n_2 are negative. In the same way as the case of Equation (a) we have ; from (b) $(n_1,n_2)>0,(n_1,q_1)>0,(n_1,q_1)>0 \Longrightarrow q_1<0, n_2<0, n_1<0$, from Equation (c) $(n_1,n_3)>0,(n_1,q_2)>0,(n_3,q_3)>0 \Longrightarrow q_2<0, n_3<0, n_1<0$, from Equation (d) $(n_1,q_1)>0,(n_1,q_2)>0,(q_1,q_2)>0 \Longrightarrow q_2<$

$0, q_1 < 0, n_1 < 0$, as described in Table-1. Considering that the negative indexes are virtual ones and classifying $I_d^{\{2\}}$ by focusing on these negative indexes n_1, n_2, n_3, we can get the relations of the same groups as Eqs. (27),(28) and (29).
Summarizing these situations, we have $I_d^{\{2\}} = I_d^{\{n_2,n_3\}} + I_d^{\{n_2,q_1\}} + I_d^{\{n_3,q_2\}} + I_d^{\{q_1,q_2\}}$ for the negative n_1, $I_d^{\{2\}} = I_d^{\{n_1,n_3\}} + I_d^{\{n_1,q_1\}}$ for the negative n_2, and $I_d^{\{2\}} = I_d^{\{n_1,n_2\}} + I_d^{\{n_1,q_1\}}$ for the negative n_3. It is surprising that these results are perfectly consistent with the results in the case of the classification by the convergent condition of Appell function $F_4(a, b; c, c'; x, y)$.

4 Concluding Remarks

In this paper we introduced negative dimensional technique and extended this idea from positive dimension until negative dimension using the concept of negative power differentiation in mathematics.[10] We proposed a new idea that we can do analytic continuation to all dimensions from negative until positive, and made a new formula to calculate Feynman propagators. Utilizing this idea and the new formula we calculated Feynman massive one loop bubble diagram. I think our idea and the new formula would be true considering the fact that results of the calculations are consistent with ones of the usual standard calculations.
The advantages of our idea and the calculation method are

(i) The calculations are simple and do not have any approximations.

(ii) We can express the results by hypergeometric series, which is useful to calculate numerically.

(iii) We can decide the domains of the power indexes in the expansions from the physical and mathematical constraints. From this fact we can draw out new physical and mathematical aspects of Feynman propagator.

The disadvantages are

(i) Cauchy's integral theorem is unstable because we made a new formula, exploiting the concept that negative power differentiation means positive power integration .

(ii) The domains of the indexes in Taylor expansion and multinomial expansions are slightly unclear.

(iii) It is indefinite whether the exchange of the integral order concerning momentum k and parameter complex variables z_i is possible or not.

Hereafter we have to verify that we can obtain accurate results by applying our idea to the calculations of more complex propagator diagrams. Furthermore we need to more scrupulously establish the domain of indexes in Taylor series expansion and multinomial expansions.

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