Parallel Algorithm for Constructing a Spanning Tree on a Certain Class of Circle Trapezoid Graphs

Hirotoshi HONMA[1] Yoko NAKAJIMA[1]

Abstract: Given a simple graph G with n vertices and m edges. The spanning tree problem is to find a spanning tree for a given graph G. This problem has many applications, such as electric power systems, computer network design and circuit analysis. For a simple graph, the spanning tree problem can be solved in $O(\log n)$ time with $O(n+m)$ processors on the CRCW PRAM. In general, it is known that more efficient parallel algorithms can be developed by restricting classes of graphs. In this paper, we shall propose a parallel algorithm which run $O(\log n)$ time with $O(n/\log n)$ processors on the EREW PRAM for constructing on certain class of circle trapezoid graphs.

Key words: Design and Analysis of Algorithms, Parallel Algorithms, Circle Trapezoid Graphs, Spanning Tree.

1 Introduction

Given a simple connected graph G with n vertices. The *spanning tree problem* is to find a tree that connects all the vertices of G. Generally, there exist a number of different spanning trees in a connected graph. Let T be a tree with n vertices. Then the following statements are equivalent [1]:

(i) T contains no cycles, and has $n-1$ edges,
(ii) T is connected, and has $n-1$ edges,
(iii) T is connected, and each edge is a bridge,
(iv) any two vertices of T are connected by exactly one path,
(v) T contains no cycles, but the addition of any new edge creates exactly one cycle.

The spanning tree problem have applications, such as electric power systems, computer network design and circuit analysis [1]. A spanning tree can be found in $O(n+m)$ time using, for example, the depth-first search or breadth-first search. In recent years, a large number of studies have been made to parallelize known sequential algorithms. For simple graphs, Chin et al. presented that the spanning forest can be found in $O(\log^2 n)$ time using $O(n^2/\log^2 n)$ processors [2]. Moreover, for a connected graph, Klein and Stein demonstrated that a spanning tree can be found in $O(\log n)$ time with $O(n+m)$ processors on the CRCW (Concurrent Read Concurrent Write) PRAM (Parallel Random Access Machine) [3].

In general, it is known that more efficient algorithms can be developed by restricting classes of graphs. For instance, Wang et al. proposed an optimal parallel algorithm for constructing a spanning tree on *permutation graphs* that runs in $O(\log n)$ time using $O(n/\log n)$ processors on the EREW (Exclusive Read Exclusive Write) PRAM [4]. Wang et al. proposed optimal parallel algorithms for some problems including the spanning tree problem on *interval graphs* that can be executed in $O(\log n)$ time with $O(n/\log n)$ processors on the EREW PRAM [5]. Bera et al. presented an optimal parallel algorithms for finding a spanning tree on *trapezoid graphs* that takes in $O(\log n)$ time using $O(n/\log n)$ processors on the EREW PRAM [6]. In addition, Honma et al. developed parallel algorithms for finding a spanning tree on *circular permutation graphs* [7] and *circular trapezoid graphs* [8]. Both of them take in $O(\log n)$ time using $O(n/\log n)$ processors on the EREW PRAM.

Felsner et al. first introduced *circle trapezoid graphs* [9]. They also provided an $O(n^2)$ time algorithm for solving maximum independent set problem and $O(n^2 \log n)$ time algorithm for solving maximum clique problem. Recently, Lin showed that circle trapezoid graphs are superclasses of trapezoid graphs [10].

In this study, we propose a parallel algorithm for spanning tree problem on a certain class of circle trapezoid graphs. It can run in $O(\log n)$ time with $O(n/\log n)$ processors on the EREW PRAM.

2 Preliminaries

2.1 Circle trapezoid model and graph

We first illustrate the circle trapezoid model before defining the circle trapezoid graph. There is a unit circle C such that the consecutive integer i, $1 \le i \le 4n$ are assigned clockwise on the circumference (n is the number of circle trapezoids). Consider nonintersecting two arcs $A' = [a_i b_i]$ and $A'' = [c_i d_i]$ along the circumference of C. The point b_i (resp., d_i) is

[1]Department of Creative Engineering, National Institute of Technology, Kushiro College

the last point encountered when traversing A' (resp., A'') clockwise. A circle trapezoid CT_i is the region in a circle C that lies between two non-crossing chords $\langle a_i d_i \rangle$ and $\langle b_i c_i \rangle$. Without loss of generality, each circle trapezoid CT_i has four corner points a_i, b_i, c_i, d_i, and all corner points are distinct. We assume that circle trapezoids are labeled in increasing order of their corner points a_i's, i.e., $CT_i < CT_j$ if $a_i < a_j$. The geometric representation described above is called the *circle trapezoid model* (CTM). Figure 1-(a) illustrates an example of CTM M with eight circle trapezoids. The circle trapezoid with $a_i > d_i$ is called *feedback circle trapezoid*. Note that there exist two feedback circle trapezoids (CT_7, CT_8) in CTM M.

We next introduce the circle trapezoid graphs. An undirected graph G is a *circle trapezoid graph* (CTG) if it can be represented by the following CTM; each vertex of the graph corresponds to a circle trapezoid in CTM, and two vertices are adjacent in G if and only if their circle trapezoids intersect [9]. Figure 1-(b) illustrates a CTG G corresponding to CTM M shown in (a). Table 1 shows the details of CTM M of Figure 1.

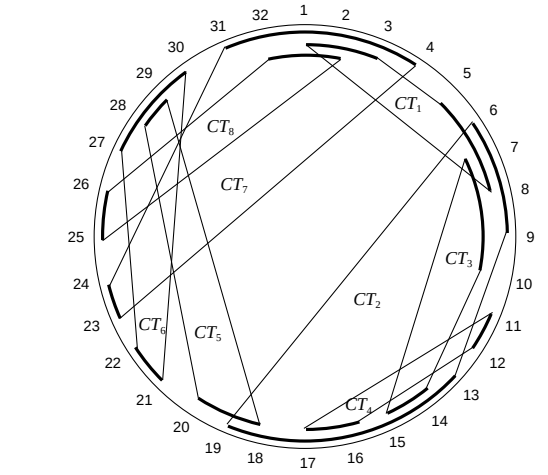
Table 1: Details of circle trapezoid model M

i	1	2	3	4	5	6	7	8
a_i	1	6	7	11	18	21	23	25
b_i	3	9	10	12	20	22	24	26
c_i	5	13	14	16	28	27	31	32
d_i	8	19	15	17	29	30	4	2

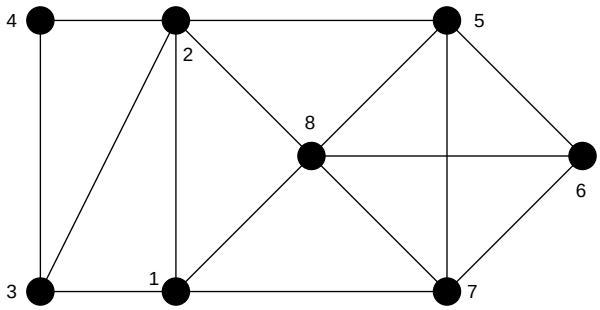
2.2 Extended circle trapezoid model

In the following, we introduce the *extended circle trapezoid model* (ECTM) constructed from a CTM for making the problem easier. We first cut CTM at point 1 on the circumference and next unroll onto the real horizontal line. Each circle trapezoid $CT_i = [a_i, b_i, c_i, d_i]$ in CTM is also changed to a pair of line segment $I_i = ([a_i, d_i], [b_i, d_i])$ called *interval pair* by executing the above process. Here, feedback circle trapezoid $CT_i = [a_i, b_i, c_i, d_i]$ in CTM is changed to interval pair $I_i = ([a_i, d_i + 4n], [b_i + 4n, d_i + 4n])$ for $a_i > b_i, c_i, d_i$. Moreover, copies I_{i-n} of I_i are created by shifting $4n$ to the left respectively, for each I_i, $1 \leq i \leq n$. Note that both interval pairs I_i and I_{i-n} in ECTM are corresponding to CT_i in CTM.

The following Algorithm 1 constructs an ECTM from a CTM. Figure 2 shows the ECTM EM constructed from the CTM M illustrated in Fig. 1. Table 2 shows the details of ECTM EM of Figure 2.



(a) Circle trapezoid model M



(b) Circle trapezoid graph G

Figure 1: Circle trapezoid model and graph

2.3 Restricted circle trapezoid model and graph

In this study, we focus and treat a certain class of circle trapezoid graphs. Graph G is a CTG corresponding to a CTM M and an ECTM EM is constructed from M by executing Algorithm 1. We consider CTM M such that the ECTM EM constructed from M satisfies that $c_i < d_j$ for two interval pairs I_i and I_j $(i < j)$ in EM. The CTM M is defined as *restricted circle trapezoid model* (rCTM). The graph corresponding to the rCTM is *restricted circle trapezoid graph* (rCTG). In this study, we will develop a parallel algorithm for spanning tree problem on rCTGs. The Figure 1 is also an example of rCTM and rCTG because $c_i < d_j$ for I_i and I_j $(i < j)$ in ECTM EM.

2.4 Other definitions

Here, some notations that form the basis of our algorithm are defined as follows.

The function nor(i) normalizes the interval pair number i in $ECTM$ within the range 1 to n, which is expressed as

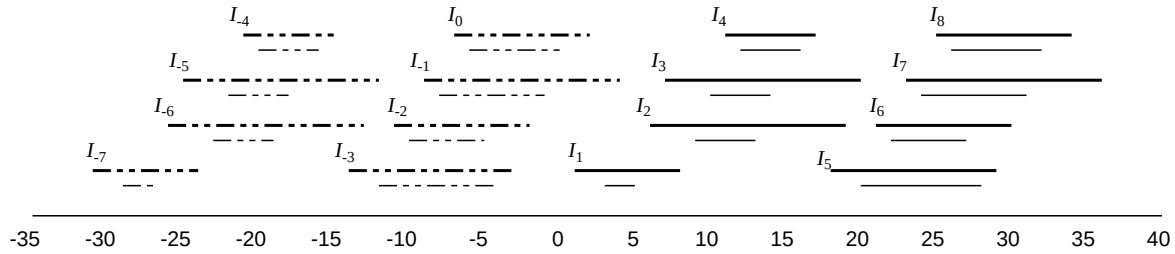


Figure 2: Extended circle trapezoid model EM

Table 2: Details of extended circle trapezoid model EM

i	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
a_i	-31	-26	-25	-21	-14	-11	-9	-7	1	6	7	11	18	21	23	25
b_i	-29	-23	-22	-20	-12	-10	-8	-6	3	9	10	12	20	22	24	26
c_i	-27	-19	-18	-16	-4	-5	-1	0	5	13	14	16	28	27	31	32
d_i	-24	-13	-17	-15	-3	-2	4	2	8	19	15	17	29	30	36	34

Construct Extended Model (CEM)

Input: Corner points $[a_i, b_i, c_i, d_i]$ of CT_i in CTM.

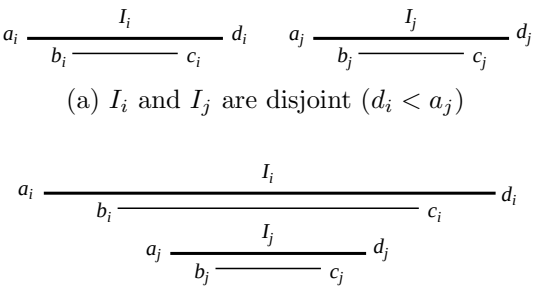
for *each non feedback circle trapezoid* CT_i **in pardo**
  Create a interval pair $I_i = ([a_i, d_i], [b_i, d_i])$;
end
for *each feedback circle trapezoid* CT_i **in pardo**
  for *each* $b_i, c_i, d_i < a_i$ **do**
    Create a interval pair
    $I_i = ([a_i, d_i + 4n], [b_i + 4n, d_i + 4n])$;
  end
end
for $1 \leq i \leq n$ **in pardo**
  Create copies I_{i-n} by shifting $4n$ to the left for I_i
end
;

$$\mathrm{nor}(i) = \begin{cases} i & \text{if } i \geq 1, \\ i + n & \text{if } i < 1. \end{cases}$$

For the example shown in Fig. 2, for $i = 4$ and $i = -5$, we have $\mathrm{nor}(4) = 4$ and $v(-5) = 3$, respectively.

The function $v_d(k)$ computes a vertex number i satisfying $d_i = k$ for a given number k on $ECTM$. For the example shown in Fig. 2, for $k = 29$ and $k = -13$, we have $v_d(29) = 5$ and $v_d(-13) = -6$ by $d_5 = 29$ and $d_{-6} = -13$, respectively. Moreover, we use $nv_d(k)$ instead of $\mathrm{nor}(v_d(k))$ for simplicity. For the example shown in Fig. 2, for $k = 29$ and $k = -13$, we have $nv_d(29) = 5$ and $nv_d(-13) = 2$ by $v_d(29) = 5$ and $v_d(-13) = -6$, respectively.

We next define $ld_i = \max\{d_{-n+1}, d_{-n+2}, \ldots, d_i\}$, for $-n + 1 \leq i \leq n - 1$, in EM_r. The details of ld_i, $v_d(d_i)$, and $nv_d(d_i)$ are shown in Table 3.



(a) I_i and I_j are disjoint ($d_i < a_j$)

(b) I_i contains I_j ($b_i < a_j$ and $c_i > d_j$)

Figure 3: Examples of disjoint and contain

3 Property of circle trapezoid graph

We describe some properties on CTGs which are useful for constructing the algorithm for spanning tree problem on rCTGs.

For two interval pairs I_i and $I_j (i < j)$ in ECTM, we say I_i and I_j are *disjoint* if $d_i < a_j$. Moreover, we say I_i *contain* I_j if $b_i < a_j$ and $d_j < c_i$. Figure 3 shows examples of the cases of disjoint and contain. The following Lemma 1 has been described in [9].

Lemma 1 *Let CT_i and CT_j $(i < j)$ be non-feedback circle trapezoids in CTM M. Moreover, ECTM EM is constructed from M. CT_i and CT_j intersect if I_i and I_j are not disjoint and I_i does not contain I_j in EM.*

The following Lemma 2 generalizes Lemma 1. This is very useful to find the edges on CTG.

Table 3: Details of extended circle trapezoid model EM

i	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
d_i	-24	-13	-17	-15	-3	-2	4	2	8	19	15	17	29	30	36	34
ld_i	-24	-13	-13	-13	-3	-2	4	4	8	19	19	19	29	30	36	-
$v_d(ld_i)$	-7	-6	-6	-6	-3	-2	-1	-1	1	2	2	2	5	6	7	-
$nv_d(ld_i)$	1	2	2	2	5	6	7	7	1	2	2	2	5	6	7	-

Lemma 2 *G is a CTG corresponding to a CTM M, and ECTM EM is constructed from M. For two interval pairs I_i, I_j $(i < j)$, an edge $(\mathrm{nor}(i), \mathrm{nor}(j))$ is in G if and only if at least one of the following conditions satisfies in EM;*

(1) $b_i > a_j$,

(2) $d_i > a_j$ and $c_i < d_j$.

(Proof) By Lemma 1, for two non-feedback circle trapezoids CT_i and CT_j do not intersect if and only if $(d_i < a_j)$ or $(b_i < a_j$ and $c_i > d_j)$ in EM. By the contra position, for two non-feedback circle trapezoids CT_i and CT_j intersect if and only if $(d_i > a_j)$ and $(b_i > a_j$ or $c_i < d_j)$ in EM. Here, $(d_i > a_j)$ and $(b_i > a_j$ or $c_i > d_j)$ is logically equal to $(d_i > a_j$ and $b_i > a_j)$ or $(d_i > a_j$ and $c_i < d_j)$.

For the condition $(d_i > a_j)$ and $(b_i > a_j)$, we have $b_i > a_j$ whenever $d_i > a_j$. Thus, $(\mathrm{nor}(i), \mathrm{nor}(j))$ is an edge of CTG G if $(b_i > a_j)$ or $(d_i > a_j$ and $c_i < d_j)$ for two interval pairs I_i and $I_j (i < j)$ in EM. □

We obtain the following Lemma 3 for restricted circle trapezoid model M_r and graph G_r.

Lemma 3 *G_r is a rCTG corresponding to a rCTM M_r, and EM_r is an extended circle trapezoid model constructed from M_r. An edge $(\mathrm{nor}(i), \mathrm{nor}(j))$ is in G_r if and only if $d_i > a_j$ for $i < j$ satisfies in the EM_r.*

(Proof) By Lemma 2, if either of conditions (1) $(b_i > a_j)$ or (2) $(d_i > a_j$ and $c_i < d_j)$ for two interval pairs I_i and $I_j (i < j)$ in EM_r, an edge $(\mathrm{nor}(i), \mathrm{nor}(j))$ is in G_r. By definition of rCTG G_r, we have $c_i < d_j$ in EM_r. Hence, condition (2) satisfies when $d_i > a_j$ holds. Moreover, condition (1) $b_i > a_j$ satisfies if $d_i > a_j$ holds because $a_i < b_i < c_i < d_i$ by the definition of interval pair. Therefore an edge $(\mathrm{nor}(i), \mathrm{nor}(j))$ is in G_r if and only if $d_i > a_j$ for $i < j$ satisfies in the EM_r. □

In Lemma 2, we have to test if $b_i > a_j$ or $(d_i > a_j$ and $c_i < d_j)$ in EM to check whether (i, j) is an edge of normal CTG G. On the other hand, by Lemma 3, we only need to check $d_i > a_j$ holds for EM_r to determine whether an edge (i, j) is in rCTG G_r.

The following Lemma 4 is core of solving this problem. An efficient algorithm can be constructed by using the following lemma.

Lemma 4 *G_r is a rCTG corresponding to a rCTM M_r, and EM_r is an extended circle trapezoid model constructed from M_r. For $1 \le i \le n$, an edge $(nv_d(ld_{i-1}), i)$ is in G_r if $ld_{i-1} > a_i$ satisfies in the EM_r.*

(Proof) By the definition, $ld_{i-1} = \max\{d_{-n+1}, d_{-n+2}, \ldots, d_{i-1}\}$. Thus, we have $v_d(ld_{i-1}) \le i - 1$. By Lemma 3, an edge $(\mathrm{nor}(i), \mathrm{nor}(j))$ is in G_r if and only if $d_i > a_j$ for $i < j$. Therefore, an edge $(nv_d(ld_{i-1}), i)$ is in G_r if $ld_{i-1} > a_i$ satisfies in the EM_r. □

4 Parallel Algorithm

In this section, we propose an algorithm for constructing a spanning tree of a connected rCTG G_r. We assume that all trapezoids in the rCTM have been sorted by corner point a in ascending order, that is, Table 1 is given as an input of our algorithm. Algorithm CST returns a spanning tree if a given graph G_r is connected. Instead of using a sophisticated technique, we propose simple parallel algorithms using only the parallel prefix computation [11] and Brent's scheduling principle [12].

Lemma 5 *After executing Step 3 of Algorithm CST, graph T is a spanning tree of CTG G_r.*

(Proof) Step 1 is a process for initialization. T is empty set and all ck_i are set to '0'. For all $-n + 1 \le i \le n - 1$, compute $ld_i := \max\{d_{-n+1}, d-n+2, \ldots, d_i\}$ using parallel prefix computation [11].

In Step 2, we set $ck_i = 1$, $1 \le i \le n$, if $ld_{i-1} > a_i$. In addition, we compute $s := \max\{\ nv_d(ld_{i-1}) \mid nv_d(ld_{i-1}) > i, ck_i = 1\}$. By Lemma 4, an edge $(nv_d(ld_{i-1}), i)$ is in G_r if $ld_{i-1} > a_i$. Thus, a vertex i that $ck_i = 1$ can have least one edge from i to other vertex $nv_d(ld_{i-1})$. Vertex s is the largest $nv_d(ld_{i-1})$ satisfying $nv_d(ld_{i-1}) > i$. For the example shown in Fig. 4, we set $ck_i = 1$ because $ld_{i-1} > a_i$,

i	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
a_i	-31	-26	-25	-21	-14	-11	-9	-7	1	6	7	11	18	21	23	25
dl_i	-24	-13	-13	-13	-3	-2	4	4	8	19	19	19	29	30	36	
ck_i									1	1	1	1	1	1	1	1

i	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
$nv_d(dl_i)$								7	1	2	2	2	5	6	7	

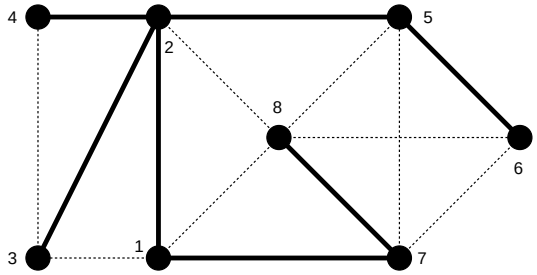


Figure 4: Example of constructed spanning tree

for $1 \leq i \leq n$. For the only case of $i = 1$, we have $nv_d(ld_{i-1}) > i$ then $s = nv_d(ld_0) = 7$.

We consider that T is added an edge form i to $v_d(ld_{i-1})$ smaller for i, $1 \leq i \leq n$. By definitions of v_d and ld, if $v_d(ld_{i-1})$ corresponds a copy of a non-feedback circle trapezoid, we have $nv_d(ld_{i-1}) < i$. On the other hand, if $v_d(ld_{i-1})$ corresponds a copy of a feedback circle trapezoid, we have $nv_d(ld_{i-1}) > i$. In the case of $\sum_{i=1}^{n} ck_i = n$, T constructed in above way is connected graph that has n vertices. Thus, T is not a tree that have exactly one cycle C. There exist some edge $(nv_d(ld_{i-1}), i)$ in C such that $nv_d(ld_{i-1})$ is feedback circle trapezoid, because a given G_r is connected. In Step 2, we obtained $s = \max\{\ nv_d(ld_{i-1}) \mid nv_d(ld_{i-1}) > i, ck_i = 1\}$. In not adding $(nv_d(ld_{s-1}), s)$ to T, we can remove a cycle C.

In Step 3, in the case of $\sum_{i=1}^{n} ck_i = n$, we add an edge $(nv_d(ld_{i-1}), i)$ to T for vertex i, $1 \leq i \leq n$, $i \neq s$. T is connected with $n-1$ vertices, that is, T is a tree. In Step 3, we consider the case of $\sum_{i=1}^{n} ck_i \neq n$. This implies $\sum_{i=1}^{n} ck_i = n - 1$. If $\sum_{i=1}^{n} ck_i < n - 1$, this means that a given G_r is disconnected, which is a contradiction to our hypothesis. Therefore, in the case of $\sum_{i=1}^{n} ck_i = n$, we add an edge $(nv_d(ld_{i-1}), i)$ to T for vertex i, $1 \leq i \leq n$, $ck_i = 1$. T is connected with $n - 1$ vertices, that is, T is a tree.

Therefore, after executing Step 3 of Algorithm CST, graph T is a spanning tree of CTG G_r. □

Figure 4 shows a spanning tree T constructed from CTG G_r by executing Algorithm CST.

In the following, we analyze the complexity of Algorithm CST.

In Step 1, an ECTM is constructed from a CTM in $O(1)$ time using $O(n)$ processors, which can be implemented in $O(\log n)$ time using $O(n/\log n)$ processors by applying Brent's scheduling principle [12]. Moreover, all rd_i are obtained in $O(\log n)$ time using $O(n/\log n)$ processors by applying parallel prefix computation [11]. In Step 2, ck_i and s are computed in $O(\log n)$ time using $O(n/\log n)$ processors by applying Brent's scheduling principle. Step 3 can also be implemented in $O(\log n)$ time using $O(n/\log n)$ processors by applying Brent's scheduling principle. In addition, Algorithm CST can be executed on an EREW PRAM because neither concurrent read nor concurrent write are necessary. Thus, we have the subsequent theorem.

Theorem 1 *Algorithm CST constructs a spanning tree of a restricted circle trapezoid graph in $O(\log n)$ time using $O(n/\log n)$ processors on EREW PRAM.* □

5 Concluding Remarks

In this paper, we presented a parallel algorithm to solve the spanning tree problem on a restricted circle trapezoid graph. This algorithm can be implemented in $O(\log n)$ time with $O(n/\log n)$ processors on an EREW PRAM computation model using only parallel prefix computation [11] and Brent's scheduling principle [12] without using a sophisticated technique. Solutions to the spanning problem have applications in electrical power provision, computer network design, circuit analysis, among others. For this reason, we think this paper is also worthy from both a theoretical and algorithmic point of view. In the future, we will continue this research by extending the results to other classes of graphs.

Construct Spanning Tree (CST)

Input: Corner points $[a_i, b_i, c_i, d_i]$ of I_i in M_r.
Output: Spanning tree T of G.

(Step 1) /* Initialization */
Construct EM from M_r using Algorithm CEM;
$T := \emptyset$;
for $1 \leq i \leq n$ **in pardo** $ck_i := 0$;
for $-n+1 \leq i \leq n-1$ **in pardo**
| $ld_i := \max\{d_{-n+1}, d-n+2, \ldots, d_i\}$;
end

(Step 2) /* Set Flag ck_i and Compute s */
for $1 \leq i \leq n$ **in pardo**
| **if** $ld_{i-1} > a_i$ **then** $ck_i := 1$;
end
$s := \max\{\ nv_d(ld_{i-1}) \mid nv_d(ld_{i-1}) > i, ck_i = 1\}$;

(Step 3) /* Construction Spanning Tree */
if $\sum_{i=1}^{n} ck_i = n$ **then**
| **for** $1 \leq i \leq n$, $i \neq s$ **in pardo**
| | $T := T \cup edge(nv_d(ld_{i-1}), i)$;
| **end**
| **return** T ;
else
| **for** $1 \leq i \leq n$, $ck_i = 0$ **in pardo**
| | $T := T \cup edge(nv_d(ld_{i-1}), i)$;
| **end**
| **return** T ;
end

References

[1] R. J. Wilson. *Introduction to Graph Theory.* Prentice Hall, 1996.

[2] F. Y. Chin, J. Lam, and I. Chen. Efficient parallel algorithms for some graph problems. *Commun. ACM*, 25:659–665, 1982.

[3] P. Klein and C. Stein. A parallel algorithm for eliminating cycle in undirected graphs. *Inf. Process. Lett.*, 34(6):307–312, 1990.

[4] Y. L. Wang, H. C. Chen, and C. Y. Lee. An $O(\log n)$ parallel algorithm for constructing a spanning tree on permutation graphs. *Inf. Process. Lett.*, 56(2):83–87, 1995.

[5] Y. L. Wang, K. C. Chiang, and M. S. Yu. Optimal algorithms for interval graphs. *J. Inf. Sci. Eng.*, 14(2):449–459, 1998.

[6] D. Bera, M. Pal, and T. K. Pal. An optimal PRAM algorithm for a spanning tree on trapezoid graphs. *J. Applied Mathematics and Computing*, 1-2:21–29, 2003.

[7] H. Honma, S. Honma, and S. Masuyama. An optimal parallel algorithm for constructing a spanning tree on circular permutation graphs. *IEICE Trans. Inf. & Syst.*, E92-D(2):141–148, 2009.

[8] H. Honma, Y. Nakajima, Y. Igarashi, and S. Masuyama. Algorithm for finding maximum detour hinge vertices of interval graphs. *IEICE Trans. Fundamentals*, E97-A(6):1365–1369, 2014.

[9] S. Felsner, R. Müller, and L. Wernisch. Trapezoid graphs and generalization, geometry and algorithms. *Discrete Appl. Math.*, 74(1):13–32, 1997.

[10] W. L. Lin. Circular and circle trapezoid graphs. *J. Sci. Eng. Tech.*, 2(2):11–17, 2006.

[11] A. Gibbons and W. Rytter. *Efficient Parallel Algorithms.* Cambridge University Press, 1988.

[12] R. P. Brent. The parallel evaluation of general arithmetic expressions. *J. ACM*, 21(2):201–206, 1974.