Negative dimensional integral technique and its extension.II

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Abstract

In the previous paper, we extended the idea of negative dimensional technique devised by I.G.Halliday and R.M.Ricotta to all dimensions using complex variable integral and fractional calculus, and showed that the single poles only are effective when we calculate Feynman integrals. In this paper we calculate more complex Feynman integrals rather than bubble Feynman diagram, and demonstrate that our new method of the calculations is true.

Key Words : Feynman integrals, Negative dimension, Hyper-geometric function, Fractional calculus

1 Introduction

The weak bosons $Z^0, W^{\pm}$ were discovered in 1983 using SPS (the super proton synchrotron) at CERN and it proved that Weinberg-Salam theory is exactly true. Hence Electromagnetic interaction and weak interaction were unified. In 2013 the Higgs bosons were detected at CERN using LHC (the large hadron collider). Recently amazing discoveries have been found at CERN. These facts provide us with information of an approach to the clarification of the origin of mass and the structure of the universe. When we want to compare the theories in gauge field theory or quantum chromodynamics to experimental data, we have to calculate Feynman propagator integrals in most cases. Dimensional regularization method founded by G.'tHooft and M.Veltman, was the most successful one as a tool of Feynman propagator calculation in quantum field theory.[1] In 1987, I.G.Halliday and R.M.Ricotta conceived an idea known as negative dimensional integral method.[2] When we calculated Feynman propagators, we proposed a new parameter transformation and a new regularization, so-called hypersurface regularization.[3-5] In the previous paper we extended the idea of negative dimension to all dimensions.[6] Using our idea and our calculation method of Feynman propagator integrals, we calculated bubble one loop Feynman integral, following the consideration of C.Anastasiou, E.W.N.Glover and C.Oleari.[7] In this paper we calculate three point Feynman integral, the so called vertex function. In Sec.2, we recapitulate the negative dimensional integral method and our idea as an extension of this method. In Sec.3 we apply our idea to three point Feynman diagram integral calculation. In Sec.4, we use our formula to the calculation of the propagator which appears when we calculate the decay width of the process $K^0_S \to 2\gamma$. It is shown that the result is completely consistent with one of the previous calculations including the sign and the normalization constant.[5] In Sec.5 we discuss the advantages and the doubtful points concerning our new idea and the new formula, and explain prospects for the future as concluding remarks.

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2 Negative dimensional integral technique and its extension

First, we review the negative dimensional integral technique, discovered by Halliday and Ricotta, and its extension to all the dimension. Hence we recall our new formula to calculate Feynman propagator integrals.[3,6]

Let us consider the following integral

$$\int d^dp \frac{(p^2)^\alpha}{(p^2+M^2)^\beta} = \pi^{\frac{d}{2}}(M^2)^{\frac{d}{2}+\alpha-\beta}\frac{\Gamma(\alpha+d/2)\Gamma(\beta-\alpha-d/2)}{\Gamma(d/2)\Gamma(\beta)}. \tag{1}$$

Now we examine the integral $\int (p^2)^\alpha d^dp$ in the case of $\beta = 0$. The result becomes 0. In order to avoid this difficulty we have to take $\alpha + \frac{d}{2} = 0$. Accordingly , because α is positive, the dimension d must be negative. This is an idea of negative dimension, proposed by Halliday and Ricotta, that means an analytic continuation to negative dimension.

The concrete calculation is as follows,

$$\begin{aligned}\int (p^2)^\alpha d^dp &= \lim_{\beta\to 0}\int \frac{(p^2)^\alpha}{(p^2+M^2)^\beta} d^dp \\ &= \lim_{\beta\to 0}\pi^{\frac{d}{2}}(M^2)^{\alpha-\beta+\frac{d}{2}}\frac{\Gamma(\alpha+d/2)\Gamma(\beta-\alpha-d/2)}{\Gamma(d/2)\Gamma(\beta)} = (-1)^\alpha \pi^{\frac{d}{2}}\Gamma(\alpha+1)\delta_{\alpha+\frac{d}{2},0},\end{aligned} \tag{2}$$

where we utilized the formula,

$$\begin{aligned}&\frac{\Gamma(\alpha+d/2)}{\Gamma(d/2)} = \left(\frac{d}{2}\right)_\alpha = (-\alpha)_\alpha \\ &= (-\alpha)(-\alpha+1)\cdots\cdots(-\alpha+\alpha-1) = (-1)^\alpha 1\times 2\times\cdots\cdots\times(\alpha-1)\times\alpha = (-1)^\alpha\Gamma(\alpha+1).\end{aligned} \tag{3}$$

Equation(2) means that it is only sufficient to replace the integral by $(-1)^\alpha\pi^{\frac{d}{2}}\Gamma(\alpha+1)\delta_{\alpha+\frac{d}{2},0}$, when having an integral such as the above type.

Next we introduce a complex parameter variable z and parametrize the above integral as the contour integral along a small circle C around the origin $z = 0$ on the complex plane. Because $\exp(-p^2z)$ is holomorphic at $z = 0$, the following equation holds

$$\begin{aligned}\int (p^2)^\alpha d^dp &= \frac{(-1)^\alpha(\alpha)!}{2\pi i}\oint_C \frac{dz}{z^{\alpha+1}}\int d^dp \exp(-p^2z) = \frac{\alpha!(-1)^\alpha}{2\pi i}\oint_C \frac{1}{z^{\alpha+1}}\left(\frac{\pi}{z}\right)^{\frac{d}{2}}dz \\ &= \frac{\alpha!(-1)^\alpha\pi^{\frac{d}{2}}}{2\pi i}\oint_C \frac{1}{z^{\alpha+1+\frac{d}{2}}}dz = (-1)^\alpha\pi^{\frac{d}{2}}\Gamma(\alpha+1)\delta_{\alpha+\frac{d}{2},0},\end{aligned} \tag{4}$$

applying Cauchy's integral theorem concerning a function with $\alpha+1$ order pole at $z = 0$. We make out that negative dimensional integral method is consistent with the concept that a single pole only is effective on the integral calculation of the complex variable parameter z.

Now we extend negative dimensional integral to all dimensional integrals in order that

$$\begin{aligned}\int \frac{d^dp}{(p^2)^\alpha} &= \frac{(-1)^\alpha(-\alpha)!}{2\pi i}\oint_C \frac{dz}{z^{1-\alpha}}\int d^dp \exp(-p^2z) = \frac{(-\alpha)!(-1)^\alpha}{2\pi i}\oint_C \frac{1}{z^{1-\alpha}}\left(\frac{\pi}{z}\right)^{\frac{d}{2}}dz \\ &= \frac{(-\alpha)!(-1)^\alpha\pi^{\frac{d}{2}}}{2\pi i}\oint_C \frac{1}{z^{1-\alpha+\frac{d}{2}}}dz = (-1)^\alpha\pi^{\frac{d}{2}}\Gamma(1-\alpha)\delta_{-\alpha+\frac{d}{2},0}\end{aligned} \tag{5}$$

holds.(See Appendix B.) In this extension we adopted the concept that $-n$ times differentiation means n times integration. In this case it is not contraditory that we take the integral domain from z to infinity.[10](See Appendix B.)

We try to calculate the following integral, utilizing our new formula, in order to examine whether our new formula holds or not. We have the following integral from Eq.(7) in Appendix C-2 of Ref. [8],

$$I = \frac{1}{i(2\pi)^d}\int d^d k \frac{1}{(m^2+2k\cdot p-k^2)^\alpha} = \frac{\Gamma(\alpha-d/2)}{(4\pi)^{\frac{d}{2}}\Gamma(\alpha)}\frac{1}{(m^2+p^2)^{\alpha-\frac{d}{2}}}. \tag{6}$$

We diagonalize the momentum k and shift it as follows $\hat{k}=k-p$. Then the process of the calculation can be shown as

$$\begin{aligned} I &= \int d^d k \frac{1}{(m^2+2kp-k^2)^\alpha} = \int d^d\hat{k}\frac{1}{(-\hat{k}^2+p^2+m^2)^\alpha} \\ &= \frac{(-\alpha)!}{2\pi i}\oint_C \frac{dz}{z^{1-\alpha}}\int d^d\hat{k}\exp\left(-\hat{k}^2+p^2+m^2\right)z = \frac{(-\alpha)!}{2\pi i}\oint_C\frac{dz}{z^{1-\alpha}}\left(\frac{\pi}{z}\right)^{\frac{d}{2}}\exp\left(p^2+m^2\right)z. \end{aligned} \tag{7}$$

Finally we can obtain the result as follows,

$$\begin{aligned} I &= (-\alpha)!\pi^{\frac{d}{2}}\sum_{n=0}^{\infty}\frac{(p^2+m^2)^n}{n!}\frac{1}{2\pi i}\oint_C\frac{dz}{z^{1-\alpha+\frac{d}{2}-n}} = (-\alpha)!\pi^{\frac{d}{2}}\sum_{n=0}^{\infty}\frac{(p^2+m^2)^n}{n!}\delta_{n+\alpha-\frac{d}{2},0} \\ &= \pi^{\frac{d}{2}}\frac{\Gamma(1-\alpha)}{\Gamma(1+d/2-\alpha)}(p^2+m^2)^{\frac{d}{2}-\alpha} = \pi^{\frac{d}{2}}(-1)^{\frac{d}{2}}\frac{\Gamma(\alpha-d/2)}{\Gamma(\alpha)}\frac{1}{(p^2+m^2)^{\alpha-\frac{d}{2}}}. \end{aligned} \tag{8}$$

This final result is consistent with one of the usual calculations except for a factor $(-1)^{\frac{d}{2}}$. In the calculation of the final term, we utilized the formula concerning Γ-function; (See Appendix A.)

$$\frac{\Gamma(1-\alpha)}{\Gamma(1+d/2-\alpha)} = \frac{\Gamma(\alpha-d/2)(-1)^{\alpha-\frac{d}{2}}}{\Gamma(\alpha)(-1)^\alpha}. \tag{9}$$

3 The application to three point Feynman propagator calculation

At this time, we will apply our idea and the new formula to the calculation of the three point massive propagator diagram described in Fig.1.

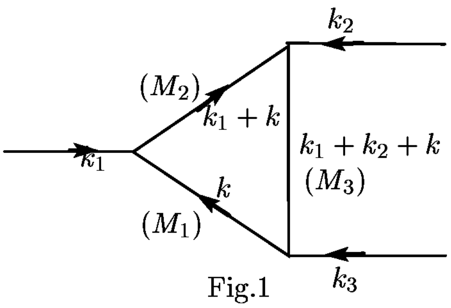

Fig.1

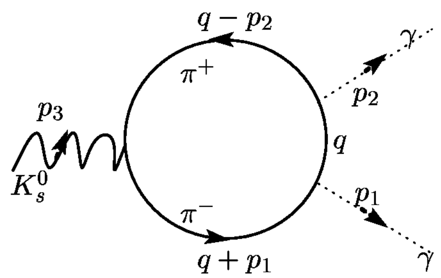

Fig.2

We have to calculate the following integral;

$$I_d^{\{3\}} = \frac{1}{i\pi^{\frac{d}{2}}}\int\frac{d^d k}{[k^2-M_1^2]^{\nu_1}[(k^2+k_1)^2-M_2^2]^{\nu_2}[(k+k_1+k_2)^2-M_3^2]^{\nu_3}}, \tag{10}$$

where we introduced the indexes ν_1,ν_2,ν_3 to generalize the denominator . We parametrize the Eq. (10) using our formula Eq.(5).
That is,

$$\begin{aligned} I_d^{\{3\}} &= \frac{1}{\pi^{\frac{d}{2}}}\frac{(-\nu_1)!(-\nu_2)!(-\nu_3)!}{(2\pi i)^3}\oint_C\frac{dz_1dz_2dz_3}{z_1^{1-\nu_1}z_2^{1-\nu_2}z_3^{1-\nu_3}} \\ &\times\int d^d\tilde{k}\exp\Big[-(\tilde{k}^2+M_1^2)z_1-((\tilde{k}+\tilde{k}_1)^2+M_2^2)z_2-((\tilde{k}+\tilde{k}_3)^2+M_3^2)z_3\Big], \end{aligned} \tag{11}$$

where we transformed the momentum k in Minkovski space to $\tilde{k}$ in Euclidean space. Now we execute the diagonalization of $\tilde{k}$ and after that we take the sifting from $\tilde{k}$ to $\hat{k}$. Moreover we exchange the integral order concerning variables $\hat{k}$ and $z_i (i = 1, 2, 3)$, and carry out the $\hat{k}$ integration first,

$$\begin{aligned} I_d^{\{3\}} &= \frac{1}{i\pi^{\frac{d}{2}}} \frac{(-\nu_1)!(-\nu_2)!(-\nu_3)!}{(2\pi i)^3} \oint_C \frac{dz_1 dz_2 dz_3}{z_1^{1-\nu_1} z_2^{1-\nu_2} z_3^{1-\nu_3}} \int d^d\hat{k} \\ &\quad \times \exp\Big[-\rho\hat{k}^2\Big] \exp\Big[-\frac{\tilde{k}_1^2 z_1 z_2 + \tilde{k}_2^2 z_2 z_3 + \tilde{k}_3^2 z_1 z_3}{z_1 + z_2 + z_3} - (M_1^2 z_1 + M_2^2 z_2 + M_3^2 z_3)\Big] \\ &= \frac{(-\nu_1)!(-\nu_2)!(-\nu_3)!}{(2\pi i)^3} \oint_C \frac{dz_1 dz_2 dz_3}{z_1^{1-\nu_1} z_2^{1-\nu_2} z_3^{1-\nu_3}} \frac{1}{\rho^{\frac{d}{2}}} \\ &\quad \times \exp\Big[\frac{k_1^2 z_1 z_2 + k_2^2 z_2 z_3 + k_3^2 z_3 z_1}{\rho} - (M_1^2 z_1 + M_2^2 z_2 + M_3^2 z_3)\Big], \end{aligned} \tag{12}$$

where

$$\hat{k} = \tilde{k} + \frac{\tilde{k}_1 z_2 - \tilde{k}_3 z_3}{\rho}, \quad \rho = \sum_{i=1}^{3} z_i. \tag{13}$$

Next we expand the integrand to the polynomial, utilizing Taylor expansion and multinomial expansion formulas .
That is,

$$\begin{aligned} I_d^{\{3\}} &= \frac{1}{(2\pi i)^3} \oint_C \Big(\Pi_{i=1}^3 dz_i\Big) \Big(\frac{\Gamma(1-\nu_1)\Gamma(1-\nu_2)\Gamma(1-\nu_3)}{z_1^{1-\nu_1} z_2^{1-\nu_2} z_3^{1-\nu_3}}\Big) \rho^{-\frac{d}{2}} \\ &\quad \times \sum_{l_i, m_i = 0; i=1,2,3}^{\infty} \Big(\frac{k_1^2 z_2 z_3}{\rho}\Big)^{l_1} \frac{1}{\Gamma(l_1+1)} \Big(\frac{k_2^2 z_3 z_1}{\rho}\Big)^{l_2} \frac{1}{\Gamma(l_2+1)} \Big(\frac{k_3^2 z_1 z_2}{\rho}\Big)^{l_3} \frac{1}{\Gamma(l_3+1)} \\ &\quad \times (-M_1^2 z_1)^{m_1} \frac{1}{\Gamma(m_1+1)} (-M_2^2 z_2)^{m_2} \frac{1}{\Gamma(m_2+1)} (-M_3^2 z_3)^{m_3} \frac{1}{\Gamma(m_3+1)} \end{aligned} \tag{14}$$

Futhermore, we can obtain

$$\begin{aligned} I_d^{\{3\}} &= \frac{1}{(2\pi i)^3} \oint_C \Big(\Pi_{i=1}^3 dz_i\Big) \sum_{l_i, m_i = 0; i=1,2,3}^{\infty} \Big(\Pi_{i=1}^3 \frac{\Gamma(1-\nu_i)}{\Gamma(l_i+1)\Gamma(m_i+1)}\Big) \\ &\quad \times \rho^{-d/2 - l_1 - l_2 - l_3} z_1^{\nu_1 - 1 + l_2 + l_3 + m_1} z_2^{\nu_2 - 1 + l_1 + l_3 + m_2} z_3^{\nu_3 - 1 + l_1 + l_2 + m_3} \\ &\quad \times (k_1^2)^{l_1} (k_2^2)^{l_2} (k_3^2)^{l_3} (M_1^2)^{m_1} (M_2^2)^{m_2} (M_3^2)^{m_3}, \end{aligned} \tag{15}$$

and expanding $(z_1 + z_2 + z_3)^{-d/2 - l_1 - l_2 - l_3}$ by using the multipolynomial formula,

$$\begin{aligned} I_d^{\{3\}} &= \frac{1}{(2\pi i)^3} \oint_C \Big(\Pi_{i=1}^3 dz_i\Big) \sum_{l_i, m_i = 0; i=1,2,3}^{\infty} \sum_{p_i; i=1,2,3} \Big(\Pi_{i=1}^3 \frac{\Gamma(1-\nu_i)}{\Gamma(l_i+1)\Gamma(m_i+1)\Gamma(p_i+1)}\Big) \\ &\quad \times \Gamma(p_1+p_2+p_3+1) z_1^{\nu_1 - 1 + l_2 + l_3 + m_1 + p_1} z_2^{\nu_2 - 1 + l_1 + l_3 + m_2 + p_2} z_3^{\nu_3 - 1 + l_1 + l_2 + m_3 + p_3} \\ &\quad \times (k_1^2)^{l_1} (k_2^2)^{l_2} (k_3^2)^{l_3} (M_1^2)^{m_1} (M_2^2)^{m_2} (M_3^2)^{m_3}. \end{aligned} \tag{16}$$

Finally, we practice the contour integrals concerning complex variables z_1, z_2, z_3, respectively, along the small circles around the origins of z_1, z_2, z_3 on the complex planes. We can obtain the following result as the single poles only are effective;

$$\begin{aligned} I_d^{\{3\}} &= \sum_{l_i, m_i = 0; i=1,2,3}^{\infty} \sum_{p_i; i=1,2,3} \Big(\Pi_{i=1}^3 \frac{\Gamma(1-\nu_i)}{\Gamma(l_i+1)\Gamma(m_i+1)\Gamma(p_i+1)}\Big) \\ &\quad \times \Gamma(p_1+p_2+p_3+1) \delta_{\nu_1 + l_2 + l_3 + m_1 + p_1, 0} \delta_{\nu_2 + l_1 + l_3 + m_2 + p_2, 0} \delta_{\nu_3 + l_1 + l_2 + m_3 + p_3, 0} \\ &\quad \times (k_1^2)^{l_1} (k_2^2)^{l_2} (k_3^2)^{l_3} (M_1^2)^{m_1} (M_2^2)^{m_2} (M_3^2)^{m_3}. \end{aligned} \tag{17}$$

The constraint conditions are as follows,

$$l_2 + l_3 + m_1 + p_1 = -\nu_1 \tag{18}$$
$$l_1 + l_3 + m_2 + p_2 = -\nu_2 \tag{19}$$
$$l_1 + l_2 + m_3 + p_3 = -\nu_3 \tag{20}$$
$$p_1 + p_2 + p_3 = -l_1 - l_2 - l_3 - \frac{d}{2}. \tag{21}$$

By putting

$$l_1 + l_2 + l_3 = l \quad \text{and} \quad m_1 + m_2 + m_3 = m, \tag{22}$$

we can solve six equatins on arbitrary two or three variables from nine variables $l_i, m_i, p_i (i = 1, 2, 3)$ Therefore we can write down the specific functinal shapes by using hypergeometric function although it is tedious calculation. To demonstrate that our idea and the new formula are true, we would like to calculate the propagator diagram which we treated with the estimate of the decay width in the decay $K_0^S \longrightarrow 2\gamma$ as drawn in Fig.2.[5]

4 The application to the decay process $K_S^0 \to 2\gamma$

The equation to calculate is

$$I_d^{\{3\}} = \frac{1}{i\pi^{\frac{d}{2}}} \int \frac{d^d q}{[q^2 - \mu^2]^{\nu_1} [(q + p_1)^2 - \mu^2]^{\nu_2} [(q - p_2)^2 - \mu^2]^{\nu_3}}. \tag{23}$$

The process of the evaluation is as follows;

$$\begin{aligned} I_d^{\{3\}} &= \frac{(-\nu_1)!(-\nu_2)!(-\nu_3)!}{(2\pi i)^3} \oint_C \frac{dz_1 dz_2 dz_3}{z_1^{1-\nu_1} z_2^{1-\nu_2} z_3^{1-\nu_3}} \frac{1}{\rho^{\frac{d}{2}}} \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\frac{z_2 z_3 s_1}{\rho} - \rho\mu^2 \Big)^n \\ &= \frac{(-\nu_1)!(-\nu_2)!(-\nu_3)!}{(2\pi i)^3} \oint_C \frac{dz_1 dz_2 dz_3}{z_1^{1-\nu_1} z_2^{1-\nu_2} z_3^{1-\nu_3}} \frac{1}{\rho^{\frac{d}{2}}} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\{l,m\}} \frac{n!}{l!m!} \Big(\frac{z_2 z_3 s_1}{\rho} \Big)^l \Big(-\rho\mu^2 \Big)^m \end{aligned} \tag{24}$$

where we put M_1^2, M_2^2, M_3^2 into $M_1^2 = M_2^2 = M_3^2 = \mu^2$ (the pion mass squared), $p_3^2 = (p_1 + p_2)^2 = 2p_1 \cdot p_2 = s_1$(the incoming Kaon mass squared) owing to $p_1^2 = p_2^2 = 0$(the emitted real photon mass squared 0).

$$I_d^{\{3\}} = \frac{(-\nu_1)!(-\nu_2)!(-\nu_3)!}{(2\pi i)^3} \oint_C \frac{dz_1 dz_2 dz_3}{z_1^{1-\nu_1} z_2^{1-\nu_2} z_3^{1-\nu_3}} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\{l,m\}} \frac{n!}{l!m!} z_2^l z_3^l s_1^l (-\mu^2)^m \rho^{m-l-d/2} \tag{25}$$

The next processes are the same as in the case of Equation (14) and (15).
Namely we can obtain

$$\begin{aligned} I_d^{\{3\}} &= \frac{(-\nu_1)!(-\nu_2)!(-\nu_3)!}{(2\pi i)^3} \sum_{n=0}^{\infty} \sum_{\{l,m\}} \sum_{\{a,b,c\}} \frac{(m-l-d/2)!}{l!m!a!b!c!} s_1^l (-\mu^2)^m \\ &\quad \times \oint_C \frac{dz_1 dz_2 dz_3}{z_1^{1-\nu_1-a} z_2^{1-\nu_2-l-b} z_3^{1-\nu_3-l-c}}, \end{aligned} \tag{26}$$

and

$$\begin{aligned} I_d^{\{3\}} &= (-\nu_1)!(-\nu_2)!(-\nu_3)! \sum_{n=0}^{\infty} \sum_{\{l,m\}} \sum_{\{a,b,c\}} \frac{(m-l-d/2)!}{l!m!a!b!c!} s_1^l (-\mu^2)^m \\ &\quad \times \delta_{1-\nu_1-a,1} \cdot \delta_{1-\nu_2-l-b,1} \cdot \delta_{1-\nu_3-l-c,1}. \end{aligned} \tag{27}$$

The constraint conditions are as follows;

$$l+m=n,\ \ a+b+c=m-l-d/2,\ \ \nu_1+a=0,\ \ \nu_2+l+b=0,\ \ \nu_3+l+c=0, \tag{28}$$

$$a=-\nu_1, \quad n=d/2-\sigma, \tag{29}$$

$$b=-\nu_2-l, \quad c=-\nu_3-l, \quad m=d/2-\sigma-l, \tag{30}$$

where we put σ to $\sigma=\nu_1+\nu_2+\nu_3$.
Now we should pay attention to what a and n are constant.
Hence, making use of the formula $\Gamma(n+1)=n!$ and other formulas, we have

$$\begin{aligned} I_d^{\{3\}} &= (-\nu_1)!(-\nu_2)!(-\nu_3)!\sum_{\{l\}}\frac{(-\sigma-2l)!(s_1)^l(-\mu^2)^{d/2-\sigma-l}}{l!(d/2-\sigma-l)!(-\nu_1)!(-\nu_2-l)!(-\nu_3-l)!} \\ &=\Gamma(1-\nu_2)\Gamma(1-\nu_3)(-\mu^2)^{\frac{d}{2}-\sigma} \\ &\times\sum_{\{l\}}\frac{\Gamma(1-\sigma)(1-\sigma)_{-2l}}{l!\Gamma(1+d/2-\sigma)(1+d/2-\sigma)_{-l}\Gamma(1-\nu_2)(1-\nu_2)_{-l}\Gamma(1-\nu_3)(1-\nu_3)_{-l}}\Big(-\frac{s_1}{\mu^2}\Big)^l \\ &=(-\mu^2)^{\frac{d}{2}-\sigma}\frac{\Gamma(1-\sigma)}{\Gamma(1+d/2-\sigma)}\sum_{\{l\}}\frac{(-1)^{2l}(\sigma-d/2)_l(\nu_2)_l(\nu_3)_l}{l!(\sigma)_{2l}(-1)^l(-1)^l(-1)^l}\Big(-\frac{s_1}{\mu^2}\Big)^l. \end{aligned} \tag{31}$$

Exploiting the following formulas, (See Appendix A.)[9]

$$(a)_{2n}=\Big(\frac{a}{2}\Big)_n\Big(\frac{a}{2}+\frac{1}{2}\Big)_n 2^{2n} \tag{32}$$

$$\Big(\sigma\Big)_{2l}=\Big(\frac{\sigma}{2}\Big)_l\Big(\frac{\sigma}{2}+\frac{1}{2}\Big)_l 4^l \tag{33}$$

$$\frac{\Gamma(1-\sigma)\Gamma(\sigma)}{\Gamma(1+d/2-\sigma)\Gamma(\sigma-d/2)}=\frac{(-1)^{\sigma-\frac{d}{2}}}{(-1)^{\sigma}}, \tag{34}$$

we can obtain the final result as follows,

$$I_d^{\{3\}}=(\mu^2)^{d/2-\sigma}\frac{\Gamma(\sigma-d/2)}{(-1)^{\sigma}\Gamma(\sigma)}\sum_{\{l\}}\frac{(\sigma-d/2)_l(\nu_2)_l(\nu_3)_l}{l!(\sigma/2)_l(\sigma/2+1/2)_l}\Big(\frac{s_1}{4\mu^2}\Big)^l. \tag{35}$$

From the definition of hyper-geometric function ${}_3F_2(a,b,c;d,e;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{n!(d)_n(e)_n}x^n$, we have

$$I_d^{\{3\}}=(\mu^2)^{d/2-\sigma}\frac{\Gamma(\sigma-d/2)}{(-1)^{\sigma}\Gamma(\sigma)}{}_3F_2\Big(\sigma-\frac{d}{2},\nu_2,\nu_3;\frac{\sigma}{2},\frac{\sigma}{2}+\frac{1}{2};\frac{s_1}{4\mu^2}\Big) \tag{36}$$

In the case of $\nu_1=\nu_2=\nu_3=1$ and $d=4-2\epsilon(\epsilon\to 0)$ we can obtain the following final result,

$$I=\Big(-\frac{1}{2\mu^2}\Big){}_3F_2\Big(1,1,1;\frac{3}{2},2;\frac{s_1}{4\mu^2}\Big). \tag{37}$$

Eq.(37) is wholly consistent with Eq.(38) in Ref.[5] including the normalization factor. This fact would prove that our idea and the new formula are true.

5 Concluding Remarks

In this paper we reviewed negative dimensional technique and the extension of this idea from positive dimension until negative dimension using the concept of negative power differentiation in mathematics.[10] We proposed a new idea that we can do analytic continuation to all dimensions from negative until positive, and made a new formula to calculate Feynman propagators. Utilizing this idea and the new formula we calculated Feynman massive vertex diagram in section 3. Then, applying this idea to the calculation of the simplest propagator, which appeared in the estimate of the decay width of the process $K_0^S \longrightarrow 2\gamma$ decay, we showed that the result is wholly consistent with the result of the previous calculation in Ref.[5] including the normalization constant in section 4. We think our idea and the new formula would be true considering this fact.
The advantages of our idea and the calculation method are

(i) The calculations are simple and do not have any approximations. The single poles only are effective, expanding the integrand by exploiting Taylor expansion and multinominal expansion formulas. This simplifies the procedure of the calculation.

(ii) We can express the results by hypergeometric series, which is useful to calculate numerically. By using hypergeometric function, we can obtain more useful informations concerning physical and mathematical situations, because hypergeometric function can be analytic continuated from one region to another one and has the several integral representations.

(iii) We can decide the domains of the power indexes in the expansions from the physical and mathematical constraints. From this fact we can draw out new physical and mathematical aspects of Feynman propagator.

The disadvantages are

(i) Cauchy's integral theorem is unstable because we made a new formula utilizing the concept that negative times differentiation means positive times integration .

(ii) The domains of the indexes in Taylor expansion and multinomial expansions are slightly unclear.

Henceforth we have to examine the details of the formula we obtained in section 3 by considering the threshold constraints , energy momentum conservation, on shell and off shell constraints on the inner and outer legs etc. When we calculated the propagator concerning the decay width of $K_0^S \to 2\gamma$,we adopted pion mass μ as the mass with the inner line of momentum q, but at large we will have to select more general mass M or a function $f(s_1)$.

Appendix A

(1) The proof of $(A)_{-n} = \frac{(-1)^n}{(1-A)_n}$

$$(A)_n = A{\cdot}(A+1){\cdot}(A+2){\cdot}(A+3){\cdot}\cdots{\cdot}(A+n-1) = \frac{\Gamma(A)}{\Gamma(A+n)} \tag{38}$$

$$(A)_{-n} = \frac{\Gamma(A-n)}{\Gamma(A)}, \text{and } \Gamma(A)\Gamma(1-A) = \frac{\pi}{\sin \pi A} \tag{39}$$

$$(1-A)_n = \frac{\Gamma(1-A+n)}{\Gamma(1-A)} \tag{40}$$

$$\Gamma(A-n)\Gamma(1-A+n) = \frac{\pi}{\sin \pi(A-n)} = \frac{\pi}{(-1)^n \sin \pi A} = \frac{\Gamma(A)\Gamma(1-A)}{(-1)^n} \tag{41}$$

$$\Gamma(A-n)\Gamma(1-A+n) = \frac{\Gamma(A)\Gamma(1-A)}{(-1)^n} \tag{42}$$

$$\frac{\Gamma(A-n)}{\Gamma(A)} = \frac{(-1)^n\Gamma(1-A)}{\Gamma(1-A+n)}, \text{ hence } (A)_{-n} = \frac{(-1)^n}{(1-A)_n} \tag{43}$$

(2) The relation $\frac{\Gamma(n)\Gamma(1-n)}{\Gamma(m)\Gamma(1-m)} = \frac{(-1)^m}{(-1)^n}$ with integers m, n.

$$\frac{\Gamma(n)\Gamma(1-n)}{\Gamma(m)\Gamma(1-m)} = \lim_{\epsilon\to 0}\frac{\sin\pi(m+\epsilon)}{\sin\pi(n+\epsilon)} = \lim_{\epsilon\to 0}\frac{\pi\cos\pi(m+\epsilon)}{\pi\cos\pi(n+\epsilon)} = \frac{(-1)^m}{(-1)^n} \tag{44}$$

Appendix B

Let us explain more precisely the extension of negative dimension technique.
If $f(z)$ is analytic everywhere in the complex z plane, $\frac{f(z)}{(z-a)^k}$ has k order pole at $z=a$. Therefore from residue theorem we have

$$\begin{aligned}
&\oint_C \frac{f(z)}{(z-a)^k}\,dz \\
&= 2\pi i \mathrm{Res}\Big[z=a;\frac{f(z)}{(z-a)^k}\Big] = 2\pi i\frac{1}{(k-1)!}\Big[\frac{d^{k-1}}{dz^{k-1}}\Big((z-a)^k\frac{f(z)}{(z-a)^k}\Big)\Big]_{z=a} \\
&= 2\pi i\frac{1}{(k-1)!}\Big[\frac{d^{k-1}}{dz^{k-1}}f(z)\Big]_{z=a},
\end{aligned} \tag{45}$$

and hence

$$\Big[\frac{d^{k-1}}{dz^{k-1}}f(z)\Big]_{z=a} = \frac{(k-1)!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^k}\,dz, \tag{46}$$

where the contour C is an arbitrary circle which has the center at $z=a$. When we take $a=0$,we can obtain as follows;

$$\Big[\frac{d^{k-1}}{dz^{k-1}}f(z)\Big]_{z=0} = \frac{(k-1)!}{2\pi i}\oint_C \frac{f(z)}{z^k}\,dz. \tag{47}$$

We apply equation(47) to calculation of propagator.
That is, as $f(z) = \exp(-p^2 z)$ and putting $k-1=\alpha$,

$$\Big[\frac{d^\alpha}{dz^\alpha}f(z)\Big]_{z=0} = \frac{\alpha!}{2\pi i}\oint_C \frac{f(z)}{z^{\alpha+1}}\,dz, \tag{48}$$

and

$$\begin{aligned}
&\int d^dp(p^2)^\alpha = (-1)^\alpha\int d^dp\Big[\frac{d^\alpha}{dz^\alpha}\exp(-p^2z)\Big]_{z=0} = (-1)^\alpha\int d^dp\frac{\alpha!}{2\pi i}\oint_C\frac{\exp(-p^2z)}{z^{\alpha+1}}\,dz \\
&= \frac{(-1)^\alpha\alpha!}{2\pi i}\oint_C dz\frac{1}{z^{\alpha+1}}\int d^dp\exp(-p^2z) = \frac{(-1)^\alpha\alpha!}{2\pi i}\oint_C dz\frac{1}{z^{\alpha+1}}\Big(\frac{\pi}{z}\Big)^{\frac{d}{2}} \\
&= \frac{(-1)^\alpha\alpha!}{2\pi i}\pi^{\frac{d}{2}}\oint_C\frac{1}{z^{\alpha+\frac{d}{2}+1}}\,dz = (-1)^\alpha\Gamma(\alpha+1)\pi^{\frac{d}{2}}\delta_{\alpha+\frac{d}{2},0}.
\end{aligned} \tag{49}$$

Next considering an idea that $-\alpha$ times differentiation means α times integration, we get the following calculation as the extension of negative dimension technique;

$$\begin{aligned}
&\int\frac{d^dp}{(p^2)^\alpha} = (-1)^\alpha\int d^dp\Big[\int_\infty^z\cdots\cdots\int_\infty^z\exp(-p^2z)\,dz\cdots dz\Big]_{z=0} \\
&= (-1)^\alpha\int d^dp\Big[\frac{d^{-\alpha}}{dz^{-\alpha}}\exp(-p^2z)\Big]_{z=0},
\end{aligned} \tag{50}$$

and

$$\left[\frac{d^{-\alpha}}{dz^{-\alpha}}f(z)\right]_{z=0} = \frac{(-\alpha)!}{2\pi i}\oint_C \frac{f(z)}{z^{1-\alpha}}\,dz, \tag{51}$$

and hence

$$\begin{aligned}
&\int \frac{d^dp}{(p^2)^\alpha} = (-1)^\alpha \int d^dp \frac{(-\alpha)!}{2\pi i}\oint_C \frac{1}{z^{1-\alpha}}\exp\left(-p^2 z\right)dz \\
&= \frac{(-1)^\alpha(-\alpha)!}{2\pi i}\oint_C \frac{1}{z^{1-\alpha}}\,dz \int d^dp \exp\left(-p^2 z\right) = \frac{(-1)^\alpha(-\alpha)!}{2\pi i}\oint \frac{1}{z^{1-\alpha}}\left(\frac{\pi}{z}\right)^{\frac{d}{2}} dz \\
&= \frac{(-1)^\alpha(-\alpha)!}{2\pi i}\pi^{\frac{d}{2}}\oint_C \frac{1}{z^{1+\frac{d}{2}-\alpha}}\,dz = \frac{(-1)^\alpha(-\alpha)!}{2\pi i}\pi^{\frac{d}{2}}2\pi i\delta_{\frac{d}{2}-\alpha,0} \\
&= (-1)^\alpha\Gamma(1-\alpha)\pi^{\frac{d}{2}}\delta_{\frac{d}{2}-\alpha,0}.
\end{aligned} \tag{52}$$

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