On fractional integration and differentiation

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Abstract

The fractional calculus has been investigated by many physicists and mathematicians. In these days a lot of solutions for the troubles in the theories of fractional calculus could be gradually clarified and even more the fractional calculus has been adopted in the various fields of physics and mathematics. In this paper we review the histories and the main theories of fractional integration and differentiation. We introduce a fractional harmonic oscillator theory as an application of fractional calculus.

Key Words : Fractional calculus, Fractional harmonic oscillator

1 Introduction

In the previous, paper we made use of a concept that minus n-order differentiation means n-hold integration, so called, that an integration is an inverse operation of differentiation, and moreover generalized Cauchy's residue theorem in function of complex variable by using this idea[1]. When we consider the relation between differentiation and integration as an unified concept of n-order, it is not so simple that we transform a differentiation to an integration, because the differentiation is local, but the integration is nonlocal and has an integral domain (upper limit and lower limit) or integral constants. Although there are these difficult problems, it is possible that we extend the order n (natural numbers) to an arbitrary real number α in the calculus. This is a concept of fractional calculus. Recently fractional calculus was exploited to make the mechanism of some physical phenomena clear, and many good results were obtained. Fractional calculus operation becomes nonlocal, therefore its operation is more complex. Owing to its nonlocal property, we can estimate the influences of some physical phenomena which happened in the past, as they say, memory effects. It is astonishing that the damping effects are included intrinsically, applying the fractional calculus to solve the differential equation of harmonic oscillation[2]-[4].

In section 2, we tell the historical background of fractional calculus. In section 3, we explain the important formulas on the fractional integration. In section 4, we mention the famous formulas of the fractional differentiation[5][6]. In section 5, we describe a fractional harmonic oscillator theory and demonstrate its thought-provoking feature. As concluding remarks, we discuss the merits and demerits of the fractional calculus in section 6.

2 Historical background

From the beginning of the development in the calculus there existed an idea that we should consider the integration and differentiation of fractional order. In those days this concept was by

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no means paid attention to. But Leipniz mentioned this idea in a letter to L'Hospital in 1695.

The earliest systematic investigations seem to have been made in the beginning and middle of the 19th century by Liouville, Riemann, Holmgren, Euler etc. The first application of the fractional calculus to a natural phenomenon was realized by Abel. He discovered in 1823 that the integral equation for the tautochrone can be solved completely by using semiderivative formulation. In 1920 Heaviside introduced fractional differentiation in his investigation of transmission line theory. But his idea and articles were disreputable extremely among the scientists at that time. Recently the fractional calculus is becoming to be paid attention to by scientists in several fields. It is known that Raspini deduced in 2000 an SU(3) symmetric wave equation, which turned out to be fractional nature[7][8]. Závada has generalized Raspini's result[9]. Furthermore in 2005 Hermann derived a mass formula, which can find successfully the ground state masses of the charmonium by using fractional calculus[10]. It is noticed that in 2006 Goldfain suggested the dynamic unification of boson and fermion fields using fractional spin and the close connection between spin and topological properties of space-time utilizing fractional calculus. Moreover he attempted to build a field theory on the basis of fractional differential and integral operators (complex quantum field theory)[11].

3 fractional integration

Let us consider the following integration of n-fold[5][6],

$$ {}_aI^n f(x) = \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_1} f(x_0)dx_0 \cdots dx_{n-1}. \tag{1} $$

We can rewrite this integration using Cauchy's integral theorem as follows;

$$ {}_aI^n f(x) = \frac{1}{(n-1)!} \int_a^x (x-\xi)^{n-1} f(\xi)d\xi. \tag{2} $$

Now we may consider the two cases

$$ {}_aI_+^n f(x) = \frac{1}{\Gamma(n)} \int_a^x (x-\xi)^{n-1} f(\xi)d\xi \tag{3} $$

and

$$ {}_bI_-^n f(x) = \frac{1}{\Gamma(n)} \int_x^b (\xi-x)^{n-1} f(\xi)d\xi \tag{4} $$

with $n \in \mathbb{N}$.

The first of these two equations is valid for $x > a$, and the second for $x < b$. To distinguish the two cases, we assign $+$ and $-$ symbols to these cases. These formulas can be extended to fractional case considering the analytic continuation of a gamma function $\Gamma(n)$.

That is,

$$ {}_aI_+^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi)d\xi, \tag{5} $$

$$ {}_bI_-^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\xi-x)^{\alpha-1} f(\xi)d\xi, \tag{6} $$

where α is an arbitrary positive real number.

On the one hand Loiuvlle defined the following integrals by setting $a = -\infty$ and $b = \infty$,

$$ I_{L+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-\xi)^{\alpha-1} f(\xi)d\xi \tag{7} $$

$$ I_{L-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} (\xi-x)^{\alpha-1} f(\xi)d\xi, \tag{8} $$

on the other hand Riemann defined the integrals by doing $a=0$, $b=0$;

$$I_{R+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-\xi)^{\alpha-1} f(\xi) d\xi \tag{9}$$

$$I_{R-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{0} (\xi-x)^{\alpha-1} f(\xi) d\xi. \tag{10}$$

4 fractional differentiation

For the simple case $0<\alpha<1$ we can get Liouville's definition of a fractional derivative from Eq.(7) and Eq.(8), considering that a derivative is the inverse operation of an integral.

$$D_{L+}^{\alpha} f(x) = \frac{d}{dx} I_{L+}^{1-\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} (x-\xi)^{-\alpha} f(\xi) d\xi, \tag{11}$$

$$D_{L-}^{\alpha} f(x) = \frac{d}{dx} I_{L-}^{1-\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{+\infty} (\xi-x)^{-\alpha} f(\xi) d\xi. \tag{12}$$

where the introduction of the first order derivative is to give a clear definition concerning the initial condition on solving a differential equation.
Next we can write down Riemann definition of a fractional derivative from Eq.(9) and Eq.(10),

$$D_{R+}^{\alpha} f(x) = \frac{d}{dx} I_{R+}^{1-\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{0}^{x} (x-\xi)^{-\alpha} f(\xi) d\xi \tag{13}$$

$$D_{R-}^{\alpha} f(x) = \frac{d}{dx} I_{R-}^{1-\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{0} (\xi-x)^{-\alpha} f(\xi) d\xi \tag{14}$$

Moreover Caputo defined a integral by putting the first order derivative into the integration.
From Eq.(11) and Eq.(12) we can obtain Loiuvlle-Caputo definition of a differentiation as follows;

$$D_{LC+}^{\alpha} f(x) = I_{L+}^{1-\alpha} \frac{df(x)}{d\xi} = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} (x-\xi)^{-\alpha} \frac{df(\xi)}{d\xi} d\xi \tag{15}$$

$$D_{LC-}^{\alpha} f(x) = I_{L-}^{1-\alpha} \frac{df(x)}{d\xi} = \frac{1}{\Gamma(1-\alpha)} \int_{x}^{+\infty} (x-\xi)^{-\alpha} \frac{df(\xi)}{d\xi} d\xi. \tag{16}$$

As $a=b=0$ Caputo himself defined a derivative as follows;

$$D_{C+}^{\alpha} f(x) = I_{R+}^{1-\alpha} \frac{df(x)}{d\xi} = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{x} (x-\xi)^{-\alpha} \frac{df(\xi)}{d\xi} d\xi \tag{17}$$

$$D_{C-}^{\alpha} f(x) = I_{R-}^{1-\alpha} \frac{df(x)}{d\xi} = \frac{1}{\Gamma(1-\alpha)} \int_{x}^{0} (\xi-x)^{-\alpha} \frac{df(\xi)}{d\xi} d\xi \tag{18}$$

Next we can extend the first order derivative to n-order derivative[12].
That is,

$${}_{a}D_{RL+}^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{a}^{x} \frac{f(\xi)}{(x-\xi)^{\alpha-n+1}} d\xi, & (n-1<\alpha<n) \\ \frac{d^n}{dx^n} f(x), & (\alpha=n) \end{cases} \tag{19}$$

$${}_{b}D_{RL-}^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{x}^{b} \frac{f(\xi)}{(\xi-x)^{\alpha-n+1}} d\xi, & (n-1<\alpha<n) \\ \frac{d^n}{dx^n} f(x), & (\alpha=n) \end{cases} \tag{20}$$

$${}_{a}D_{C+}^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} \frac{f(\xi)^{(n)}}{(x-\xi)^{\alpha-n+1}} d\xi, & (n-1<\alpha<n) \\ \frac{d^n}{dx^n} f(x), & (\alpha=n) \end{cases} \tag{21}$$

$$
{}_bD^{\alpha}_{C-}f(x)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_x^b\frac{f(\xi)^{(n)}}{(\xi-x)^{\alpha-n+1}}d\xi, & (n-1<\alpha<n)\\ \frac{d^n}{dx^n}f(x), & (\alpha=n)\end{cases} \tag{22}
$$

where we can take the lower limit a and the upper limit b to 0 or $\pm\infty$, too.
There is a relation between D^{α}_{C} and D^{α}_{RL};

$$
D^{\alpha}_{C}f(x)=D^{\alpha}_{RL}f(x)-\sum_{k=0}^{n-1}\frac{x^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(0). \tag{23}
$$

Now we prove this relation.

Proof

At first we expand $f(x)$ to Taylor series at $x=0$.

$$
f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}x^k+\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}x^k+R_n(x), \tag{24}
$$

where

$$
R_n(x)=\sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=\int_0^x\frac{f^{(n)}(\xi)}{(n-1)!}(x-\xi)^{n-1}d\xi, \tag{25}
$$

because

$$
\begin{aligned}
R_n(x)&=\int_0^x\frac{f^{(n)}(\xi)}{(n-1)!}(x-\xi)^{n-1}d\xi=\Big[\frac{-f^{(n)}(\xi)}{(n-1)!n}(x-\xi)^n\Big]_0^x+\frac{1}{n!}\int_0^x f^{(n+1)}(\xi)(x-\xi)^n d\xi\\
&=\frac{f^{(n)}(0)}{n!}x^n+\frac{1}{n!}\Big[-\frac{1}{n+1}f^{(n+1)}(\xi)(x-\xi)^{(n+1)}\Big]_0^x+\frac{1}{(n+1)!}\int_0^x f^{(n+2)}(\xi)(x-\xi)^{n+1}d\xi\\
&=\cdots\cdots\cdots=\sum_{k=n}^{\infty}\frac{1}{k!}f^{(k)}(0)x^k
\end{aligned} \tag{26}
$$

Therefore

$$
R_n(x)=\frac{1}{\Gamma(n)}\int_0^x f^{(n)}(\xi)(x-\xi)^{n-1}d\xi=I^nf^{(n)}(x), \tag{27}
$$

where we know that I^n denotes a standard n-fold integral from Cauchy's integral theorem.
Next using Eq.(19) as $a=0$, Eq.(24) and Eq.(27), we can calculate as follows,

$$
\begin{aligned}
D^{\alpha}_{RL}f(x)&=D^{\alpha}_{RL}\Big(\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}x^k+R_n\Big)=\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}D^{\alpha}_{RL}x^k+D^{\alpha}_{RL}R_n\\
&=\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x\frac{\xi^k}{(x-\xi)^{\alpha+1-n}}d\xi+D^{\alpha}_{RL}I^nf^{(n)}(x).
\end{aligned} \tag{28}
$$

Furthermore we evaluate the integral in the first term of Eq.(28), making use of beta function $B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt$.
Using the transformation of integral variable $\xi/x=y,\ d\xi=xdy,$ the integral domain becomes from 0 to 1.

$$
\begin{aligned}
&\int_0^x\xi^k(x-\xi)^{n-\alpha-1}d\xi=\int_0^x x^k\Big(\frac{\xi}{x}\Big)^k x^{n-\alpha-1}\Big(1-\frac{\xi}{x}\Big)^{n-\alpha-1}d\xi\\
&=x^{k+n-\alpha}\int_0^1 y^k(1-y)^{n-\alpha-1}dy=x^{k+n-\alpha}B(k+1,n-\alpha)\\
&=x^{k+n-\alpha}\frac{\Gamma(k+1)\Gamma(n-\alpha)}{\Gamma(k+n-\alpha+1)}
\end{aligned} \tag{29}
$$

The first term of Eq.(28) becomes as follows;

$$\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x\frac{\xi^k}{(x-\xi)^{\alpha+1-n}}d\xi \tag{30}$$
$$=\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{\Gamma(k+1)}\frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(k+1)\Gamma(n-\alpha)}{\Gamma(k+n-\alpha+1)}x^{k-\alpha}=\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{\Gamma(k+n-\alpha+1)}x^{k-\alpha}.$$

The second term of Eq.(28) is as follows;

$$D_{RL}^{\alpha}I^n f^{(n)}(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x\frac{I^n f^{(n)}(\xi)}{(x-\xi)^{\alpha+1-n}}d\xi$$
$$=\frac{1}{\Gamma(n-\alpha)}\int_0^x(n-\alpha-1)(n-\alpha-2)\cdots(-\alpha)\frac{I^n f^{(n)}(\xi)}{(x-\xi)^{\alpha+1}}d\xi$$
$$=\frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)}\int_0^x\frac{I^n f^{(n)}(\xi)}{(x-\xi)^{\alpha+1}}d\xi=\frac{1}{\Gamma(-\alpha)}\int_0^x\{I^n f^{(n)}(\xi)\}(x-\xi)^{-\alpha-1}d\xi$$
$$=I^{-\alpha}\Big[\Big(I^n\Big)f^{(n)}(x)\Big]=I^{n-\alpha}f^{(n)}(x)$$
$$=\frac{1}{\Gamma(n-\alpha)}\int_0^x\frac{f^{(n)}(\xi)}{(x-\xi)^{\alpha+1-n}}d\xi=D_C^{\alpha}f(x) \tag{31}$$

Therefore we can obtain the following relation;

$$D_{RL}^{\alpha}f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{\Gamma(k+n-\alpha+1)}x^{k-\alpha}+D_C^{\alpha}f(x) \tag{32}$$

When we have $f^{(k)}(0)=0$, $(k=0,1,\cdots,n-1)$, $D_{RL}^{\alpha}=D_C^{\alpha}$ holds though $D_{RL}^{\alpha}f(x)$ is not equal to $D_C^{\alpha}f(x)$ generally.

5 Application of fractional calculus to harmonic oscillator

Let us consider that we solve the equation of motion of a harmonic oscillation as an application of fractional calculus.
At first we adopt the standard equation of motion as follows;

$$\frac{d^2}{dt^2}x(t)+\omega_0^2x(t)=f(t), \tag{33}$$

where ω_0 is the natural frequency and $f(t)$ is a forced oscillation function.
The initial condition is assumed $\dot{x}(0)=x(0)=0$ for simplicity.
Next multiplying $t-\tau$ to the both sides of Eq.(33) and integrating them from 0 to t, we can obtain an integral equation including the initial conditions.
That is,

$$\int_0^t(t-\tau)\ddot{x}(\tau)d\tau+\int_0^t(t-\tau)\omega_0^2x(\tau)d\tau=\int_0^t(t-\tau)f(\tau)d\tau,$$
$$\Big[(t-\tau)\dot{x}(\tau)\Big]_0^t+\int_0^t\dot{x}(\tau)d\tau+\int_0^t(t-\tau)\omega_0^2x(\tau)d\tau=\int_0^t(t-\tau)f(\tau)d\tau,$$
$$\Big[x(\tau)\Big]_0^t+\int_0^t(t-\tau)\omega_0^2x(\tau)d\tau=\int_0^t(t-\tau)f(\tau)d\tau,$$
$$x(t)=-\omega_0^2\int_0^t(t-\tau)x(\tau)d\tau+\int_0^t(t-\tau)f(\tau)d\tau. \tag{34}$$

As the right hand side of Eq.(34) forms a convolution integral, we can calculate the Laplace transformation of Eq.(34) easily,

$$x(t) = -\omega_0^2\Big[t * x(t)\Big] + \Big[t * f(t)\Big], \tag{35}$$

$$X(s) = -\omega_0^2(\mathcal{L}t)X(s) + (\mathcal{L}t)F(s), \tag{36}$$

$$X(s) = \frac{F(s)}{s^2 + \omega_0^2}, \tag{37}$$

with $\mathcal{L}x(t) = X(s)$ and $\mathcal{L}f(t) = F(s)$. The symbol * denotes a convolution integral relation. Taking the inverse transformation of Eq.(37) yields the solution $x(t)$ of Eq.(33) as follows;

$$x(t) = \mathcal{L}^{-1}\Big(\frac{F(s)}{s^2 + \omega_0^2}\Big) = \Big(\frac{1}{\omega_0}\mathcal{L}^{-1}\frac{\omega_0}{s^2 + \omega_0^2}\Big) * \mathcal{L}^{-1}F(s) = \int_0^t \frac{1}{\omega_0}\sin\omega_0(t-\tau)f(\tau)d\tau \tag{38}$$

The response $x(t)$ for a sinusoidal forced function $f(t) = A\sin\omega t$ becomes

$$\begin{aligned} x(t) &= \int_0^t \frac{1}{\omega_0}\sin\omega_0(t-\tau)f(\tau)d\tau = \int_0^t A\sin\omega\tau\frac{\sin\omega_0(t-\tau)}{\omega_0}d\tau \\ &= \frac{A\omega\sin\omega_0 t}{\omega_0(\omega_0^2 - \omega^2)} + \frac{A\sin\omega t}{\omega_0^2 - \omega^2}. \end{aligned} \tag{39}$$

Now we introduce a generalized n times derivative harmonic oscillator equation perfunctorily,

$$\frac{d^n}{dt^n}x(t) + \omega_0^n x(t) = f(t) \quad (n \in \mathbb{N}). \tag{40}$$

We assume the initial conditions as follows;

$$\frac{d^{n-1}x}{dt^{n-1}}(0) = \frac{d^{n-2}x}{dt^{n-2}}(0) = \cdots = x(0) = 0. \tag{41}$$

The integral equation satisfying these initial conditions can be sought for in the same way as Eq.(34).
That is,

$$\int_0^t (t-\tau)^{n-1}\frac{d^n}{d\tau^n}x(\tau)d\tau + \int_0^t (t-\tau)^{n-1}\omega_0^n x(\tau)d\tau = \int_0^t (t-\tau)^{n-1}f(\tau)d\tau, \tag{42}$$

$$x(t) = -\frac{\omega_0^n}{\Gamma(n)}\int_0^t (t-\tau)^{n-1}x(\tau)d\tau + \frac{1}{\Gamma(n)}\int_0^t (t-\tau)^{n-1}f(\tau)d\tau. \tag{43}$$

Extending the natural number $n \in \mathbb{N}$ to an arbitrary real number $\alpha \in \mathbb{R}$, we obtain a fractional harmonic oscillator differential equation;

$$\frac{d^\alpha}{dt^\alpha}x(t) + \omega_0^\alpha x(t) = f(t). \tag{44}$$

Furthermore considering an analytic continuation from n to α , the integral equation becomes

$$\begin{aligned} x(t) &= -\frac{\omega_0^\alpha}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}x(\tau)d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau \\ &= -\frac{\omega_0^\alpha}{\Gamma(\alpha)}\Big[x(t) * (t^{\alpha-1})\Big] + \frac{1}{\Gamma(\alpha)}\Big[f(t) * (t^{\alpha-1})\Big]. \end{aligned} \tag{45}$$

This formula is the same as the fractional integral according to Riemann definition;

$$_R I_+^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\xi)^{\alpha-1}f(\xi)d\xi \tag{46}$$

The domain of α is considered $1 < \alpha \leq 2$. We can ensure easily that Eq.(45) agrees with Eq.(34) when we take $\alpha = 2$.
Applying the Laplace transform to both side of Eq.(45) yields

$$X(s) = -\omega_0^\alpha \frac{X(s)}{s^\alpha} + \frac{F(s)}{s^\alpha}. \tag{47}$$

Therefore we have

$$X(s) = F(s)\Big(\frac{1}{s^\alpha + \omega_0^\alpha}\Big). \tag{48}$$

The solution of Eq.(48) is

$$x(t) = \mathcal{L}^{-1}X(s) = \int_0^t f(\tau)g(t-\tau)d\tau \tag{49}$$

with

$$\mathcal{L}^{-1}\Big[\frac{1}{s^\alpha + \omega_0^\alpha}\Big] = g(t), \text{ and } \mathcal{L}^{-1}F(s) = f(t). \tag{50}$$

The generalized Mittag-Leffler function is defined as,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{51}$$

The Laplace transformation of $t^{\beta-1}E_{\alpha,\beta}(-at^\alpha)$ becomes

$$\int_0^\infty \exp(-st)t^{\beta-1}E_{\alpha,\beta}(-at^\alpha)dt = \frac{s^{-\beta}}{1+as^{-\alpha}}. \tag{52}$$

Putting $\alpha = \beta$, we have

$$x(t) = \int_0^t f(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\omega_0^\alpha(t-\tau)^\alpha)d\tau. \tag{53}$$

Taking $f(t) = A\sin\omega t$ as a forced function, we have

$$x(t) = A\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\omega_0^\alpha(t-\tau)^\alpha)\omega\tau E_{2,2}(-\omega^2\tau^2)d\tau, \tag{54}$$

because we can express $\sin\omega t$ using a Mittag-Leffler function as follows;

$$tE_{2,2}(-\omega^2t^2) = t\sum_{k=0}^{\infty}\frac{(-w^2t^2)^k}{\Gamma(2k+2)} = \frac{1}{\omega}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(\omega t)^{2k+1} = \frac{\sin\omega t}{\omega}. \tag{55}$$

Taking Laplace transformation of Eq.(54) yields

$$\begin{aligned} X(s) &= A\omega\mathcal{L}\Big(t^{\alpha-1}E_{\alpha,\alpha}(-\omega_0^\alpha t^\alpha)\Big)\mathcal{L}\Big(tE_{2,2}(-\omega^2t^2)\Big) \\ &= A\omega\Big(\frac{s^{-\alpha}}{1+\omega_0^\alpha s^{-\alpha}}\Big)\Big(\frac{s^{-2}}{1+\omega^2s^{-2}}\Big) = A\omega\Big(\frac{1}{s^\alpha+\omega_0^\alpha}\Big)\Big(\frac{1}{s^2+\omega^2}\Big). \end{aligned} \tag{56}$$

Performing the inverse Laplace transform of Eq.(56), we can get the response $x(t)$ as follows;

$$x(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{A\omega\exp(st)}{(s^2+\omega^2)(s^\alpha+\omega_0^\alpha)}ds. \tag{57}$$

Adopting the Brownwich contour as shown in Fig.1, Eq.(57) can be evaluated as the sum of two contributions, $x_1(t)$ and $x_2(t)$.

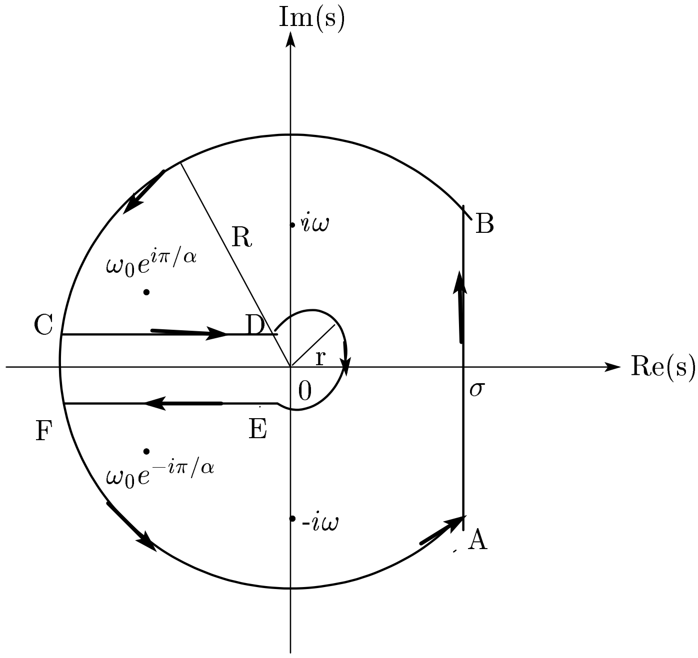

Fig.1 Brownwich contour for the integral

The function $x_1(t)$ comes from the integral along the course BCDEFA, and $x_2(t)$ comes from the four residues in the Brownwich contour. It is known that the contributions from the arcs DE, FA and BC become zero in the limits $r \to 0$ and $R \to \infty$. Evaluating the line integral along the Brownwich contour, we obtain

$$x(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} X(s)\exp(st)ds \tag{58}$$
$$= -\frac{1}{2\pi i}\int_{\text{lineCD}} X(s)\exp(st)ds - \frac{1}{2\pi i}\int_{\text{lineEF}} X(s)\exp(st)ds + \sum_i \text{Residue(i)}$$
$$= x_1(t) + x_2(t).$$

After the simple calculation concerning complex variable function, we have

$$x_1(t) = \int_0^\infty \exp(-rt) K_\alpha(r, \omega_0^\alpha) dr, \tag{59}$$

with

$$K_\alpha(r, \omega_0^\alpha) = \frac{A\omega \sin \pi\alpha}{\pi(r^2+\omega^2)(r^{2\alpha} + 2r^\alpha \omega_0^\alpha \cos \pi\alpha + \omega_0^{2\alpha})}. \tag{60}$$

We know that the value of above integration becomes zero when t tends to ∞.
Next we calculate the integration of $x_2(t)$, using the residue theorem of complex variable function . The poles in the Brownwich contour are at $s = \pm i\omega$ and $s = \omega_0 \exp(\pm i\pi/\alpha)$ under the condition $|\arg(s)| \le \pi$. The procedure of the residue calculations can be performed as follows;

$$x_2(t) = \text{Res}[s = \pm i\omega] + \text{Res}[s = \omega_0 \exp(\pm i\pi/\alpha)] = x_2'(t) + x_2"(t), \tag{61}$$

$$x_2'(t) = \text{Res}[s = i\omega] + \text{Res}[s = -i\omega] = \Big[A\omega\frac{\exp(st)}{(s+i\omega)(s^\alpha+\omega_0^\alpha)}\Big]_{s=i\omega} + \Big[A\omega\frac{\exp(st)}{(s-i\omega)(s^\alpha+\omega_0^\alpha)}\Big]_{s=-i\omega}$$

$$
\begin{aligned}
&= A\omega e^{i\omega t}\frac{-\omega^\alpha \sin(\pi\alpha/2) - i(\omega^\alpha\cos(\pi\alpha/2)+\omega_0^\alpha)}{2\omega(\omega^{2\alpha}\sin^2(\pi\alpha/2)+(\omega^\alpha\cos(\pi\alpha/2)+\omega_0^\alpha)^2)} \\
&\quad + A\omega e^{-i\omega t}\frac{-\omega^\alpha \sin(\pi\alpha/2) + i(\omega^\alpha\cos(\pi\alpha/2)+\omega_0^\alpha)}{2\omega(\omega^{2\alpha}\sin^2(\pi\alpha/2)+(\omega^\alpha\cos(\pi\alpha/2)+\omega_0^\alpha)^2)} \\
&= A\frac{\omega^\alpha\sin(\omega t-\pi\alpha/2)+\omega_0^\alpha\sin\omega t}{\omega^{2\alpha}+\omega_0^{2\alpha}+2\omega^\alpha\omega_0^\alpha\cos(\pi\alpha/2)}, \qquad (62)
\end{aligned}
$$

$$
\begin{aligned}
x_2"(t) &= \mathrm{Res}[s=\omega_0 e^{\frac{\pi}{\alpha}i}] + \mathrm{Res}[s=\omega_0 e^{-\frac{\pi}{\alpha}i}] \\
&= \Big[A\omega\frac{\exp(st)}{(s^2+\omega^2)\frac{d}{ds}(s^\alpha+\omega_0^\alpha)}\Big]_{s=\omega_0 e^{\frac{\pi}{\alpha}i}} + \Big[A\omega\frac{\exp(st)}{(s^2+\omega^2)\frac{d}{ds}(s^\alpha+\omega_0^\alpha)}\Big]_{s=\omega_0 e^{-\frac{\pi}{\alpha}i}} \qquad (63) \\
&= e^{\omega_0 t\exp(\frac{\pi}{\alpha}i)}\frac{A\omega}{(\omega_0^2 e^{\frac{2\pi}{\alpha}i}+\omega^2)\alpha\omega_0^{\alpha-1}e^{\frac{\pi}{\alpha}(\alpha-1)i}} + e^{\omega_0 t\exp(-\frac{\pi}{\alpha}i)}\frac{A\omega}{(\omega_0^2 e^{-\frac{2\pi}{\alpha}i}+\omega^2)\alpha\omega_0^{\alpha-1}e^{-\frac{\pi}{\alpha}(\alpha-1)i}} \\
&= A\omega\Big(e^{\omega_0 t\exp(\frac{\pi}{\alpha}i)}\tilde{D}_1 + e^{\omega_0 t\exp(-\frac{\pi}{\alpha}i)}\tilde{D}_2\Big), \qquad (64)
\end{aligned}
$$

where we put as follows,

$$
\begin{cases}
\tilde{D}_1 = \frac{\omega_0^2\cos\frac{\pi}{\alpha}(\alpha+1)+\omega^2\cos\frac{\pi}{\alpha}(\alpha-1)-i\big(\omega_0^2\sin\frac{\pi}{\alpha}(\alpha+1)+\omega^2\sin\frac{\pi}{\alpha}(\alpha-1)\big)}{\alpha\omega_0^{\alpha-1}\big(\omega_0^4+\omega^4+2\omega_0^2\omega^2\cos\frac{2\pi}{\alpha}\big)} \\
\tilde{D}_2 = \frac{\omega_0^2\cos\frac{\pi}{\alpha}(\alpha+1)+\omega^2\cos\frac{\pi}{\alpha}(\alpha-1)+i\big(\omega_0^2\sin\frac{\pi}{\alpha}(\alpha+1)+\omega^2\sin\frac{\pi}{\alpha}(\alpha-1)\big)}{\alpha\omega_0^{\alpha-1}\big(\omega_0^4+\omega^4+2\omega_0^2\omega^2\cos\frac{2\pi}{\alpha}\big)}.
\end{cases} \qquad (65)
$$

Therefore we have

$$
\begin{aligned}
x_2"(t) &= \frac{2A\omega\exp(\omega_0 t\cos(\pi/\alpha))}{\alpha\omega_0^{\alpha-1}(\omega_0^4+\omega^4+2\omega_0^2\omega^2\cos(2\pi/\alpha))} \qquad (66) \\
&\quad\times\Big[\omega_0^2\cos(\omega_0 t\sin\frac{\pi}{\alpha}-\frac{\pi}{\alpha}(\alpha+1)) + \omega^2\cos(\omega_0 t\sin\frac{\pi}{\alpha}-\frac{\pi}{\alpha}(\alpha-1))\Big].
\end{aligned}
$$

Finally we can obtain

$$
\begin{aligned}
x_2(t) = x_2'(t) + x_2"(t) &= A\Big[\frac{\omega_0^\alpha\sin\omega t+\omega^\alpha\sin(\omega t-\pi\alpha/2)}{\omega_0^{2\alpha}+\omega^{2\alpha}+2\omega_0^\alpha\omega^\alpha\cos(2\pi/\alpha)}\Big] \\
&+\frac{2A\omega}{\alpha\omega_0^{\alpha-1}}\exp(\omega_0 t\cos(\pi/\alpha))\Big[\frac{\omega^2\cos M+\omega_0^2\cos N}{\omega^4+\omega_0^4+2\omega^2\omega_0^2\cos(2\pi/\alpha)}\Big], \qquad (67)
\end{aligned}
$$

with

$$
M = (\omega_0 t\sin(\pi/\alpha) + \pi(1-\alpha)/\alpha), \quad N = (\omega_0 t\sin(\pi/\alpha) - \pi(1+\alpha)/\alpha). \qquad (68)
$$

After some algebra,we can get

$$
x_2(t) = A_1\sin(\omega t-\delta) + A_2\exp(-\gamma t)\cos(\omega_0 t\sin(\pi/\alpha)-\phi), \qquad (69)
$$

with

$$
\begin{cases}
A_1 = \frac{A}{(\omega_0^{2\alpha}+\omega^{2\alpha}+2\omega_0^\alpha\omega^\alpha\cos(\pi\alpha/2))^{1/2}} \\
\delta = \arctan\Big[\frac{\omega^\alpha\sin(\pi\alpha/2)}{\omega_0^\alpha+\omega^\alpha\cos(\pi\alpha/2)}\Big] \\
A_2 = \frac{2A\omega}{\alpha\omega_0^{\alpha-1}(\omega_0^4+\omega^4+2\omega_0^2\omega^2\cos(\pi\alpha/2))^{1/2}} \\
\gamma = -\omega_0\cos(\pi/\alpha) \\
\phi = \arctan\Big[\frac{\omega_0^2\sin(\pi(1+\alpha)/\alpha)-\omega^2\sin(\pi(1-\alpha)/\alpha)}{\omega_0^2\cos(\pi(1+\alpha)/\alpha)+\omega^2\cos(\pi(1-\alpha)/\alpha)}\Big].
\end{cases} \qquad (70)
$$

The first term of Eq.(69) is a standard harmonic oscillation term. As the conditin $\cos(\pi/\alpha) < 0$ ($1 < \alpha \leq 2$) is true, the second term is an intrinsic damping effect term by making use of fractional calculus in spite of neither damping factors nor damping functions. It is an astonishing aspect of fractional harmonic oscillator. We naturally understand that Eq.(69) agrees with Eq.(39) perfectly by putting $\alpha = 2$.

6 Concluding remarks

In this paper we reviewed the historical aspect and the main important formulas of fractional calculus and introduced a fractional harmonic oscillator theory as an application of fractional calculus. It was surprising that an intrinsic damping term appears in the solution by exploiting fractional harmonic oscillator equation without adding any damping factors or cutoff functions.
The advantages of using fractional calculus are; (i) We can treat integration and differentiation unitedly as differintegration. (ii) We can estimate the phenomena which happened at the past as memory effects owing to the nonlocality of fractional calculus. The disadvantages are; (i) There exist many definitions with respect to differintegration and the relations among them are not perfectly clear until now. (ii) The calculations are more complex for the reason of non locality of fractional calculus. (iii) The physical meanings of fractional calculus are not completely clear.

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