The consideration of the both analytic continuations to the right and left Semicircle contours in Davydychev method calculation

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Abstract

Davydychev discovered a useful and beautiful calculation method with respect to Feynman integrals, using hypergeometric function theory and its analytic continuation. We employed the half plane where there are smaller number of singularities at the time of utilizing Davydychev method. In this paper we consider both analytic continuation to the right and left Semicircle contours. Consequently we can derive new analytic continuation formulas on hypergeometric functions.

Key Words : Hypergeometric fuction, Feynman Integrals, Dimensional regularization

1 Introduction

When we want to compare the result we obtained by using a physical theory with the experimental data in regards to a physical phenomenon, we have to calculate Feynman integrals mostly in the final stage, especially in field theory. The calculation methods have been developed and improved by many physicists over the past decades. In particular dimensional regularization which was discovered by G.'tHooft and M.Veltman is the most effective calculation method and has been utilized by many researchers.[1] While investigating the ways to find the values of Feynman integrals, we found a new parameter transformation and a calculation method.[2]-[5]

In this paper we introduce and review the Davydychev method that I think it is one of the most beautiful calculation methods for Feynman integral calculations.[6][7] Furthermore we find new analytic continuation formulas for hypergeometric functions ${}_2F_1(z)$ and $F_4(x, y)$. In section 2 we formulate the calculation of one loop type propagator with power indexes α and β. First of all we make the calculation in the case of $m_1 = 0$ and $m_2 = m$ applying the theory of hypergeometric function. Further we estimate it under another convergence condition and show that it perfectly agrees with the result of the calculation by using the analytic continuation formula concerning hypergeometric function ${}_2F_1(z)$. In section 3 we calculate the one-loop type integral in the case of $m_1 = m_2 = m$ under the several convergence conditions. Hence we can give hypergeometric function ${}_3F_2(z)$ a new analytic continuation formula which ${}_3F_2(z)$ will be established. In section 4 we evaluate the one-loop type propagator in the case of $m_1 \neq m_2$ making use of hypergeometric function $F_4(x, y)$ with two variables, so called Appell's function. And then we show that we can write down a new analytic continuation formula concerning to $F_4(x, y)$. In section 5 we discuss advantages and disadvantages of the Davydychev method by utilizing hypergeometric function. Moreover we show prospects in the future.

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2 Formulation and the calculation in the case of $m_1 = 0$ and $m_2 = m$

Let's consider the simplest one-loop type propagator integral with masses m_1, m_2. The integral can be expressed as

$$J(\alpha,\beta;m_1,m_2) = \int \frac{d^n k}{(k^2-m_1^2)^\alpha[(p-k)^2-m_2^2]^\beta}, \tag{1}$$

where $n = 4-2\epsilon$ and we introduced the power indexes α and β to keep the generality.
Now we examine the integral in the case of $m_1 = 0$ and $m_2 = m$.
That is,

$$J(\alpha,\beta;0,m) = \int \frac{d^n k}{(k^2)^\alpha[(p-k)^2-m^2]^\beta}. \tag{2}$$

Using Taylor expansion under the convergence condition $m^2/(p-k)^2 < 1$ and the definition of the hypergeometric function ${}_1F_0(\beta,z) = \sum_{n-0}^{\infty}\frac{(\beta)_n}{n!}z^n$, we obtain

$$\frac{1}{[(p-k)^2-m^2]^\beta} = \frac{1}{[(p-k)^2]^\beta}\sum_{j=0}^{\infty}\frac{1}{j!}\beta(\beta+1)\cdots(\beta+j-1)\Big(\frac{m^2}{(p-k)^2}\Big)^j$$
$$= \frac{1}{[(p-k)^2]^\beta}\sum_{j=0}^{\infty}\frac{(\beta)_j}{j!}\Big(\frac{m^2}{(p-k)^2}\Big)^j = \frac{1}{[(p-k)^2]^\beta}{}_1F_0\Big(\beta;\frac{m^2}{(p-k)^2}\Big). \tag{3}$$

The Barnes integral representation is expressed as

$${}_1F_0(\beta;z) = \frac{1}{\Gamma(\beta)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds(-z)^s\Gamma(-s)\Gamma(\beta+s) = \frac{1}{(1-z)^\beta}. \tag{4}$$

Substituting (3),(4) for (2), we get the following equation;

$$J(\alpha,\beta;0,m) = \frac{1}{\Gamma(\beta)2\pi i}\int_{-i\infty}^{i\infty}ds\Gamma(-s)\Gamma(\beta+s)(-m^2)^s\int\frac{d^nk}{(k^2)^\alpha[(p-k)^2]^{\beta+s}}$$
$$= \frac{1}{\Gamma(\beta)2\pi i}\int_{-i\infty}^{i\infty}ds\Gamma(-s)\Gamma(\beta+s)(-m^2)^sJ^{(0)}(\alpha,\beta+s), \tag{5}$$

where $J^{(0)}(\alpha,\beta+s)$ is massless propagator.
$J^{(0)}(\alpha,\beta)$ is well-known and given as follows;

$$J^{(0)}(\alpha,\beta) = J(\alpha,\beta;0,0) = \int\frac{d^nk}{(k^2)^\alpha[(p-k)^2]^\beta} = \pi^{\frac{n}{2}}i^{1-n}(p^2)^{\frac{n}{2}-\alpha-\beta}\frac{\Gamma(n/2-\alpha)\Gamma(n/2-\beta)\Gamma(\alpha+\beta-n/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-\alpha-\beta)}. \tag{6}$$

Substituting (6) for (5) yields the following integral;

$$J(\alpha,\beta;0,m) = \pi^{\frac{n}{2}}i^{1-n}(p^2)^{\frac{n}{2}-\alpha-\beta}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds\Big(-\frac{m^2}{p^2}\Big)^s\frac{\Gamma(-s)\Gamma(n/2-\alpha)\Gamma(n/2-\beta-s)\Gamma(\alpha+\beta+s-n/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-\alpha-\beta-s)}. \tag{7}$$

Replacing the variable $\frac{n}{2}-\alpha-\beta-s$ with s, we have

$$J(\alpha,\beta;0,m) = \pi^{\frac{n}{2}}i^{1-n}(-m^2)^{\frac{n}{2}-\alpha-\beta}\frac{\Gamma(n/2-\alpha)}{\Gamma(\alpha)\Gamma(\beta)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds\Big(-\frac{p^2}{m^2}\Big)^s\frac{\Gamma(-s)\Gamma(s+\alpha)\Gamma(s+\alpha+\beta-n/2)}{\Gamma(s+n/2)}. \tag{8}$$

From the property of gamma function $\Gamma(-s) = \lim_{j\to\infty}\frac{\Gamma(j+1-s)}{(j-s)(j-1-s)\cdot\cdots\cdot(1-s)(-s)}$, we understand that $\Gamma(-s)$ only in the integrand is a meromorphic function with $s = j (j = 0,1,2,\dots)$ as

simple poles in the right half plane of complex variable s.
Now we perform the contour integral of (8) on the contour C enclosing all the poles of $\Gamma(-s)$, which starts from $-iR$ on the imaginary axis and runs along the imaginary axis to iR, then goes back to the starting point along a right semicircle C_R of the radius R. Because the contour integral on this right semicircle C_R tends to 0 when $R \to 0$, what we have to do is only the residue integral by using residue theorem concerning complex variable integral as follows;

$$\mathrm{Residue[s=j;Integrand]} = \sum_{\mathrm{j}=0}^{\infty} \Big(\frac{\mathrm{p}^2}{\mathrm{m}^2}\Big)^{\mathrm{j}} \frac{\Gamma(\alpha+\mathrm{j})\Gamma(\alpha+\beta-\mathrm{n}/2+\mathrm{j})}{\mathrm{j}!\Gamma(\mathrm{n}/2+\mathrm{j})}, \tag{9}$$

and

$$\begin{aligned} J(\alpha,\beta;0,m) &= \pi^{\frac{n}{2}} i^{1-n} (-m^2)^{\frac{n}{2}-\alpha-\beta} \frac{\Gamma(n/2-\alpha)}{\Gamma(\alpha)\Gamma(\beta)} \mathrm{Residue[s=j;Integrand]} \\ &= \pi^{\frac{n}{2}} i^{1-n} (-m^2)^{\frac{n}{2}-\alpha-\beta} \frac{\Gamma(n/2-\alpha)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+j)\Gamma(\alpha+\beta-n/2+j)}{j!\Gamma(n/2+j)} \Big(\frac{p^2}{m^2}\Big)^j \\ &= \pi^{\frac{n}{2}} i^{1-n} (-m^2)^{\frac{n}{2}-\alpha-\beta} \frac{\Gamma(n/2-\alpha)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{j=0}^{\infty} \frac{\Gamma(\alpha)(\alpha)_j\Gamma(\alpha+\beta-n/2)(\alpha+\beta-n/2)_j}{j!\Gamma(n/2)(n/2)_j} \Big(\frac{p^2}{m^2}\Big)^j \\ &= \pi^{\frac{n}{2}} i^{1-n} (-m^2)^{\frac{n}{2}-\alpha-\beta} \frac{\Gamma(n/2-\alpha)\Gamma(\alpha+\beta-n/2)}{\Gamma(\beta)\Gamma(n/2)} {}_2F_1(\alpha,\alpha+\beta-n/2;n/2;p^2/m^2), \end{aligned} \tag{10}$$

where a hypergeometric function ${}_2F_1(z)$ is defined as ${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} z^n$.
This result holds under the convergence condition of hypergeometric series $p^2/m^2 < 1$.

Next closing the contour of integration on the left half plane of complex variable s, we perform the contour integral in the same way as it used to. From (8) $\Gamma(s+\alpha)$ and $\Gamma(s+\alpha+\beta-n/2)$ in the integrand have single poles in the left semicircle. Therefore we can calculate the integral by applying residue theorem. The residues of $\Gamma(s+\alpha)$ and $\Gamma(s+\alpha+\beta-n/2)$ are found because $\Gamma(s+\alpha)$ and $\Gamma(s+\alpha+\beta-n/2)$ have single poles at $s=-j-\alpha$ and $s=-j-\alpha-\beta+n/2$, respectively, from the property of gamma function.
Namely we have

$$\begin{aligned} J(\alpha,\beta;0,m) &= \pi^{\frac{n}{2}} i^{1-n} (-m^2)^{\frac{n}{2}-\alpha-\beta} \frac{\Gamma(n/2-\alpha)}{\Gamma(\alpha)\Gamma(\beta)} \\ &\times \Big(\sum_{j=0}^{\infty} \mathrm{Residue[s=-j-\alpha;Integrand]} + \sum_{\mathrm{j}=0}^{\infty} \mathrm{Residue[s=-j-(\alpha+\beta-n/2);Integrand]} \Big), \end{aligned} \tag{11}$$

where the residue calculations were given as follows;

$$\begin{aligned} &\sum_{j=0}^{\infty} \mathrm{Residue[s=-j-\alpha;Integrand]} = \Big(-\frac{\mathrm{m}^2}{\mathrm{p}^2}\Big)^{\alpha} \frac{\Gamma(\alpha)\Gamma(\beta-\mathrm{n}/2)}{\Gamma(\mathrm{n}/2-\alpha)} \sum_{\mathrm{j}=0}^{\infty} \frac{(\alpha)_{\mathrm{j}}(1+\alpha-\mathrm{n}/2)_{\mathrm{j}}}{\mathrm{j}!(1+\mathrm{n}/2-\beta)_{\mathrm{j}}} \Big(\frac{\mathrm{m}^2}{\mathrm{p}^2}\Big)^{\mathrm{j}} \\ &= \Big(-\frac{m^2}{p^2}\Big)^{\alpha} \frac{\Gamma(\alpha)\Gamma(\beta-n/2)}{\Gamma(n/2-\alpha)} {}_2F_1(\alpha,1+\alpha-n/2;1+n/2-\beta;m^2/p^2), \end{aligned} \tag{12}$$

and

$$\sum_{j=0}^{\infty} \mathrm{Residue[s=-j-(\alpha+\beta-n/2);Integrand]}$$

$$=\Big(-\frac{m^2}{p^2}\Big)^{\alpha+\beta-n/2}\frac{\Gamma(\alpha+\beta-n/2)\Gamma(n/2-\beta)}{\Gamma(n-\alpha-\beta)}\sum_{j=0}^{\infty}\frac{(\alpha+\beta-n/2)_j(1+\alpha+\beta-n)j}{j!(1+\beta-n/2)_j}\Big(\frac{m^2}{p^2}\Big)^j \tag{13}$$

$$=\Big(-\frac{m^2}{p^2}\Big)^{\alpha+\beta-n/2}\frac{\Gamma(\alpha+\beta-n/2)\Gamma(n/2-\beta)}{\Gamma(n-\alpha-\beta)}{}_2F_1(\alpha+\beta-n/2,1+\alpha+\beta-n;1+\beta-n/2;m^2/p^2),$$

where we used a formula $(a)_{-j}=(-1)^j/(1-a)_j$.[8]
The result of the contour integral becomes

$$J(\alpha,\beta;0.m)=\pi^{\frac{n}{2}}i^{1-n}(p^2)^{\frac{n}{2}-\alpha-\beta}\Bigg[\Big(-\frac{m^2}{p^2}\Big)^{\frac{n}{2}-\beta}\frac{\Gamma(\beta-n/2)}{\Gamma(\beta)}{}_2F_1(\alpha,1+\alpha-n/2;1+n/2-\beta;m^2/p^2)$$

$$+\frac{\Gamma(n/2-\alpha)\Gamma(n/2-\beta)\Gamma(\alpha+\beta-n/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-\alpha-\beta)}{}_2F_1(\alpha+\beta-n/2,1+\alpha+\beta-n;1+\beta-n/2;m^2/p^2)\Bigg].\tag{14}$$

Furthermore utilizing analytic continuation formula, we can get the same result as the residue integral,too. That is,the analytic continuation formula [9] is described as

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1(a,a-c+1;a-b+1;1/z)$$

$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1(b,b-c+1;b-a+1;1/z).\tag{15}$$

Taking the assignment $a=\alpha,\ b=\alpha+\beta-\frac{n}{2},\ c=\frac{n}{2},\ z=\frac{p^2}{m^2}$,
from this formula we can obtain the analytic continuation formula of ${}_2F_1(\alpha,\alpha+\beta-n/2;n/2;p^2/m^2)$;

$${}_2F_1(\alpha,\alpha+\beta-n/2;n/2;p^2/m^2)$$

$$=\frac{\Gamma(n/2)\Gamma(\beta-n/2)}{\Gamma(\alpha+\beta-n/2)\Gamma(n/2-\alpha)}\Big(-\frac{p^2}{m^2}\Big)^{-\alpha}{}_2F_1(\alpha,\alpha-n/2+1;n/2-\beta+1;m^2/p^2)\tag{16}$$

$$+\frac{\Gamma(n/2)\Gamma(n/2-\beta)}{\Gamma(\alpha)\Gamma(n-\alpha-\beta)}\Big(-\frac{p^2}{m^2}\Big)^{-\alpha-\beta+\frac{n}{2}}{}_2F_1(\alpha+\beta-n/2,\alpha+\beta-n+1;\beta-n/2+1;m^2/p^2)$$

Substituting (16) for (10), we can gain

$$J(\alpha,\beta;0,m)=\pi^{\frac{n}{2}}i^{1-n}(-m^2)^{\frac{n}{2}-\alpha-\beta}\frac{\Gamma(n/2-\alpha)\Gamma(\alpha+\beta-n/2)}{\Gamma(n/2)\Gamma(\beta)}$$

$$\times\Bigg[\frac{\Gamma(n/2)\Gamma(\beta-n/2)}{\Gamma(\alpha+\beta-n/2)\Gamma(n/2-\alpha)}\Big(-\frac{p^2}{m^2}\Big)^{-\alpha}{}_2F_1(\alpha,\alpha-n/2+1,n/2-\beta+1;m^2/p^2)$$

$$+\frac{\Gamma(n/2)\Gamma(n/2-\beta)}{\Gamma(\alpha)\Gamma(n-\alpha-\beta)}\Big(-\frac{m^2}{p^2}\Big)^{\alpha+\beta-\frac{n}{2}}{}_2F_1(\alpha+\beta-n/2,\alpha+\beta-n+1;\beta-n/2+1;m^2/p)\Bigg]$$

$$=\pi^{\frac{n}{2}}i^{1-n}(p^2)^{\frac{n}{2}-\alpha-\beta}\times\Bigg[\frac{\Gamma(\beta-n/2)}{\Gamma(\beta)}\Big(-\frac{m^2}{p^2}\Big)^{\frac{n}{2}-\beta}{}_2F_1(\alpha,\alpha-n/2+1;n/2-\beta+1;m^2/p^2)\tag{17}$$

$$+\frac{\Gamma(n/2-\beta)\Gamma(n/2-\alpha)\Gamma(\alpha+\beta-n/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-\alpha-\beta)}{}_2F_1(\alpha+\beta-n/2,\alpha+\beta-n+1;\beta-n/2+1;m^2/p^2)\Bigg].$$

This formula perfectly coincides with (14) which was obtained by using the contour integration on the left half plane. This fact is so surprising. It means that we can write down many kind of analytic continuation formulas by calculating Feynman propagators.

3 Calculation in the case of $m_1 = m_2 = m$

Next we consider another special case of the integral of (1) with $m_1 = m_2 = m$.
Making use of Taylor expansion, the definition of ${}_1F_0(a, z)$ and Barnes integral representation of ${}_1F_0(a, z)$ produces

$$J(\alpha, \beta; m, m) = \int \frac{d^n k}{(k^2 - m^2)^\alpha [(p-k)^2 - m^2]^\beta} = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{(2\pi i)^2}$$
$$\times \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} dsdt \Gamma(-s)\Gamma(s+\alpha)\Gamma(-t)\Gamma(t+\beta)(-m^2)^{s+t} \int \frac{d^n k}{(k^2)^{\alpha+s}[(p-k)^2]^{\beta+t}}, \quad (18)$$

and from (6)

$$J^{(0)}(\alpha+s, \beta+t) = \int \frac{d^n k}{(k^2)^{\alpha+s}[(p-k)^2]^{\beta+t}}$$
$$= \pi^{\frac{n}{2}} i^{1-n} (p^2)^{\frac{n}{2}-\alpha-\beta-s-t} \frac{\Gamma(n/2-\alpha-s)\Gamma(n/2-\beta-t)\Gamma(\alpha+\beta-n/2+s+t)}{\Gamma(\alpha+s)\Gamma(\beta+t)\Gamma(n-\alpha-\beta-s-t)}, \quad (19)$$

where $J^{(0)}(\alpha+s, \beta+t)$ is massless propagator.
Substituting (19) for(18), we obtain

$$J(\alpha, \beta; m, m) = \int \frac{d^n k}{(k^2 - m^2)^\alpha [(p-k)^2 - m^2]^\beta} = \pi^{\frac{n}{2}} i^{1-n} (p^2)^{\frac{n}{2}-\alpha-\beta} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{(2\pi i)^2} \quad (20)$$
$$\times \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} dsdt \frac{\Gamma(-s)\Gamma(-t)\Gamma(n/2-\alpha-s)\Gamma(n/2-\beta-t)\Gamma(\alpha+\beta-n/2+s+t)}{\Gamma(n-\alpha-\beta-s-t)} \Big(-\frac{m^2}{p^2}\Big)^{s+t}$$

Changing variable t to $t = \frac{n}{2} - \alpha - \beta - s - u$, adopting Barnes Formula; [8]

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s) ds = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)} \quad (21)$$

with $a = \alpha + \beta - n/2 + u, b = \alpha + u, c = 0, d = n/2 - \alpha$, and being able to integrate on s because $(-m^2/p^2)^{s+t} = (-m^2/p^2)^{n/2-\alpha-\beta-u}$ is independent of s,

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(\alpha+\beta-n/2+u+s)\Gamma(\alpha+u+s)\Gamma(-s)\Gamma(n/2-\alpha-s) ds$$
$$= \frac{\Gamma(\alpha+\beta-n/2+u)\Gamma(\beta+u)\Gamma(\alpha+u)\Gamma(n/2+u)}{\Gamma(\alpha+\beta+2u)}, \quad (22)$$

we can get the following integral;

$$J(\alpha, \beta; m, m) = \pi^{\frac{n}{2}} i^{1-n} (-m^2)^{\frac{n}{2}-\alpha-\beta} [\Gamma(\alpha)\Gamma(\beta)]^{-1}$$
$$\times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} du \Big(-\frac{p^2}{m^2}\Big)^u \frac{\Gamma(-u)\Gamma(\alpha+u)\Gamma(\beta+u)\Gamma(\alpha+\beta-n/2+u)}{\Gamma(\alpha+\beta+2u)}. \quad (23)$$

Closing the contour of integration to the right half plane and taking the right semicircle at the center of origin as an integral contour on the u variable complex plane, $\Gamma(-u)$ only has single poles with $u = j (j = 0, 1, 2 \ldots)$ in the integrand. Applying the residue theorem yields

$$J(\alpha, \beta; m, m) = \pi^{\frac{n}{2}} i^{1-n} (-m^2)^{\frac{n}{2}-\alpha-\beta} [\Gamma(\alpha)\Gamma(\beta)]^{-1} \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+j)\Gamma(\beta+j)\Gamma(\alpha+\beta-n/2+j)}{(-1)^j j! \Gamma(\alpha+\beta+2j)}$$
$$\times \Big(-\frac{p^2}{m^2}\Big)^j = \pi^{\frac{n}{2}} i^{1-n} (-m^2)^{\frac{n}{2}-\alpha-\beta} \frac{\Gamma(\alpha+\beta-n/2)}{\Gamma(\alpha+\beta)} \sum_{j=0}^{\infty} \frac{(\alpha)_j (\beta)_j (\alpha+\beta-n/2)_j}{(-1)^j j! (\alpha+\beta)_{2j}} \Big(-\frac{p^2}{m^2}\Big)^j \quad (24)$$

Exploiting the formula $(a)_{2j} = (\frac{1}{2}a)_j(\frac{1}{2}a+\frac{1}{2})_j 2^{2j}$,[8] we obtain the final result as follows;

$$J(\alpha,\beta;m,m) = \pi^{\frac{n}{2}} i^{1-n}(-m^2)^{\frac{n}{2}-\alpha-\beta}\frac{\Gamma(\alpha+\beta-n/2)}{\Gamma(\alpha+\beta)}$$
$$\times\sum_{j=0}^{\infty}\frac{(\alpha)_j(\beta)_j(\alpha+\beta-n/2)_j}{j!((1/2)(\alpha+\beta))_j((1/2)(\alpha+\beta+1))_j}\Big(\frac{p^2}{4m^2}\Big)^j = \pi^{\frac{n}{2}} i^{1-n}(-m^2)^{\frac{n}{2}-\alpha-\beta}\frac{\Gamma(\alpha+\beta-n/2)}{\Gamma(\alpha+\beta)}$$
$$\times {}_3F_2\Big(\alpha,\beta,\alpha+\beta-n/2;(1/2)(\alpha+\beta),(1/2)(\alpha+\beta+1);p^2/4m^2\Big). \tag{25}$$

Moreover we make another attempt that we close the contour of integration to the left half plane. In this case $\Gamma(\alpha+u)$, $\Gamma(\beta+u)$ and $\Gamma(\alpha+\beta-n/2+u)$ have single poles in this semicircle region with $u=-j-\alpha$, $u=-j-\beta$ and $u=-j-\alpha-\beta+n/2$,respectively, where the integer j tends from 0 to ∞ .
The residue integration on $\Gamma(\alpha+u)$ becomes

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}du\Big(-\frac{p^2}{m^2}\Big)^u\frac{\Gamma(-u)\Gamma(\alpha+u)\Gamma(\beta+u)\Gamma(\alpha+\beta-n/2+u)}{\Gamma(\alpha+\beta+2u)}$$
$$=\Big(-\frac{m^2}{p^2}\Big)^{\alpha}\Gamma(\alpha)\Gamma(\beta-n/2)\sum_{j=0}^{\infty}\Big(\frac{m^2}{p^2}\Big)^j\frac{(\alpha)_j(\beta-\alpha)_{-j}(\beta-n/2)_{-j}}{j!(\beta-\alpha)_{-2j}}$$
$$=\Big(-\frac{m^2}{p^2}\Big)^{\alpha}\Gamma(\alpha)\Gamma(\beta-n/2)\sum_{j=0}^{\infty}\Big(\frac{4m^2}{p^2}\Big)^j\frac{(\alpha)_j((1/2)(\alpha-\beta+1))_j((1/2)(2+\alpha-\beta))_j}{j!(1+\alpha-\beta)_j(1+n/2-\beta)_j} \tag{26}$$
$$=\Big(-\frac{m^2}{p^2}\Big)^{\alpha}\Gamma(\alpha)\Gamma(\beta-n/2){}_3F_2\Big(\alpha,(1/2)(\alpha-\beta+1),(1/2)(\alpha-\beta+2);1+\alpha-\beta,1+n/2-\beta;4m^2/p^2\Big),$$

and the residue one concerning $\Gamma(\beta+u)$ is just enough to change (α,β) to (β,α).
That is,

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}du\Big(-\frac{p^2}{m^2}\Big)^u\frac{\Gamma(-u)\Gamma(\alpha+u)\Gamma(\beta+u)\Gamma(\alpha+\beta-n/2+u)}{\Gamma(\alpha+\beta+2u)} = \Big(-\frac{m^2}{p^2}\Big)^{\beta}\Gamma(\beta)\Gamma(\alpha-n/2)$$
$$\times{}_3F_2(\beta,1/2(\beta-\alpha+1),1/2(\beta-\alpha+2);1+\beta-\alpha,1+n/2-\alpha;4m^2/p^2). \tag{27}$$

The residue integration on $\Gamma(\alpha+\beta-n/2+u)$ is

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}du\Big(-\frac{p^2}{m^2}\Big)^u\frac{\Gamma(-u)\Gamma(\alpha+u)\Gamma(\beta+u)\Gamma(\alpha+\beta-n/2+u)}{\Gamma(\alpha+\beta+2u)}$$
$$=\Big(-\frac{p^2}{m^2}\Big)^{\frac{n}{2}-\alpha-\beta}\sum_{j=0}^{\infty}\Big(-\frac{m^2}{p^2}\Big)^j\frac{\Gamma(\alpha+\beta-n/2+j)\Gamma(n/2-\beta-j)\Gamma(n/2-\alpha-j)}{j!(-1)^j\Gamma(n-\alpha-\beta-2j)}$$
$$=\Big(-\frac{p^2}{m^2}\Big)^{\frac{n}{2}-\alpha-\beta}\frac{\Gamma(\alpha+\beta-n/2)\Gamma(n/2-\beta)\Gamma(n/2-\alpha)}{\Gamma(n-\alpha-\beta)}$$
$$\times\sum_{j=0}^{\infty}\Big(\frac{4m^2}{p^2}\Big)^j\frac{(\alpha+\beta-n/2)_j((1/2)(\alpha+\beta-n+1))_j((1/2)(\alpha+\beta-n+2))_j}{j!(1+\alpha-n/2)_j(1+\beta-n/2)_j}$$
$$=\Big(-\frac{p^2}{m^2}\Big)^{\frac{n}{2}-\alpha-\beta}\frac{\Gamma(\alpha+\beta-n/2)\Gamma(n/2-\beta)\Gamma(n/2-\alpha)}{\Gamma(n-\alpha-\beta)} \tag{28}$$
$$\times{}_3F_2\Big(\alpha+\beta-n/2,(1/2)(\alpha+\beta-n+1),(1/2)(\alpha+\beta-n+2);1+\alpha-n/2,1+\beta-n/2;4m^2/p^2\Big).$$

Substituting (26),(27) and (28) for (23), we obtain the total result as follows;

$$J(\alpha,\beta;m,m) = \pi^{\frac{n}{2}} i^{1-n}(p^2)^{\frac{n}{2}-\alpha-\beta}$$

$$\times\Bigg[\Big(-\frac{m^2}{p^2}\Big)^{\frac{n}{2}-\beta}\frac{\Gamma(\beta-n/2)}{\Gamma(\beta)}{}_3F_2\Big(\alpha,(1/2)(\alpha-\beta+1),(1/2)(\alpha-\beta+2);1+\alpha-\beta,1+n/2-\beta;4m^2/p^2\Big)$$
$$+\Big(-\frac{m^2}{p^2}\Big)^{\frac{n}{2}-\alpha}\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}{}_3F_2\Big(\beta,(1/2)(\beta-\alpha+1),(1/2)(\beta-\alpha+2);1+\beta-\alpha,1+n/2-\alpha;4m^2/p^2\Big)$$
$$+\frac{\Gamma(\alpha+\beta-n/2)\Gamma(n/2-\alpha)\Gamma(n/2-\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-\alpha-\beta)} \tag{29}$$
$$\times{}_3F_2\Big(\alpha+\beta-n/2,(1/2)(\alpha+\beta-n+1),(1/2)(\alpha+\beta-n+2);1+\alpha-n/2,1+\beta-n/2;4m^2/p^2\Big)\Bigg].$$

We can construct a new analytic continuation formula of ${}_3F_2(z)$ as $n=4,\alpha=a,\beta=b$ and $p^2/4m^2=z$, referring to (25) and (29);

$${}_3F_2\Big(a,b,a+b-2;(1/2)(a+b),(1/2)(a+b+1);z\Big)=(-4z)^{2-a-b}\frac{\Gamma(a+b)\Gamma(2-a)\Gamma(2-b)}{\Gamma(a)\Gamma(b)\Gamma(4-a-b)}$$
$$\times{}_3F_2\Big(a+b-2,(1/2)(a+b-3),(1/2)(a+b-2);a-1,b-1;1/z\Big) \tag{30}$$
$$+(-4z)^{-b}\frac{\Gamma(a+b)\Gamma(a-2)}{\Gamma(a)\Gamma(a+b-2)}{}_3F_2\Big(b,(1/2)(b-a+1),(1/2)(b-a+2);3-a,b-a+1;1/z\Big)$$
$$+(-4z)^{-a}\frac{\Gamma(a+b)\Gamma(b-2)}{\Gamma(b)\Gamma(a+b-2)}{}_3F_2\Big(a,(1/2)(a-b+1),(1/2)(a-b+2);3-b,a-b+1;1/z\Big).$$

4 Calculation in the case of $m_1 \neq m_2$

We consider the integral in the case of $m_1\neq m_2$ from (1). Adopting the propagator calculation of massless particles (6) and utilizing Taylor expansion, the definition of hypergeometric function ${}_1F_0$ and Barnes integral representation of ${}_1F_0$, we have

$$J(\alpha,\beta;m_1,m_2)=\pi^{\frac{n}{2}}i^{1-n}(p^2)^{\frac{n}{2}-\alpha-\beta}[\Gamma(\alpha)\Gamma(\beta)]^{-1}\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}dsdt\Big(-\frac{m_1^2}{p^2}\Big)^s\Big(-\frac{m_2^2}{p^2}\Big)^t$$
$$\times\frac{\Gamma(-s)\Gamma(-t)\Gamma(n/2-\alpha-s)\Gamma(n/2-\beta-t)\Gamma(\alpha+\beta-n/2+s+t)}{\Gamma(n-\alpha-\beta-s-t)}. \tag{31}$$

Closing the contour to the right half plane on each complex sheet of s and t variable complex planes,we know there are four cases taking the singularities in s-half plane and in t-half plane from (31), that is, (i)the case of taking single poles of $\Gamma(-s)$ and $\Gamma(-t)$, (ii)the case of taking single poles of $\Gamma(-s)$ and $\Gamma(n/2-\beta-t)$, (iii)the case of considering $\Gamma(n/2-\alpha-s)$ and $\Gamma(-t)$, (iv)the case of considering $\Gamma(n/2-\alpha-s)$ and $\Gamma(n/2-\beta-t)$.
We can calculate the residues in each case, respectively.
(i) The case of $\Gamma(-s)$ and $\Gamma(-t)$
There exist single poles $s=j_1$ and $t=j_2$ $(j_1=1,2,\ldots,j_2=1,2,\ldots)$, respectively, on each of the s,t complex sheets. Therefore the residue becomes as follows;

$$\mathrm{Residue}\Big[\mathrm{s=j_1,t=j_2;Integrand}\Big]$$
$$=\sum_{j_1}^{\infty}\sum_{j_2}^{\infty}\Big(-\frac{m_1}{p^2}\Big)^{j_1}\Big(-\frac{m_2}{p^2}\Big)^{j_2}\frac{\Gamma(n/2-\alpha-j_1)\Gamma(n/2-\beta-j_2)\Gamma(\alpha+\beta-n/2+j_1+j_2)}{j_1!(-1)^{j_1}j_2!(-1)^{j_2}\Gamma(n-\alpha-\beta-j_1-j_2)}$$
$$=\frac{\Gamma(n/2-\alpha)\Gamma(n/2-\beta)\Gamma(\alpha+\beta-n/2)}{\Gamma(n-\alpha-\beta)}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\Big(\frac{m_1^2}{p^2}\Big)^{j_1}\Big(\frac{m_2^2}{p^2}\Big)^{j_2}$$
$$\times\frac{(\alpha+\beta-n/2)_{j_1+j_2}(1+\alpha+\beta-n)_{j_1+j_2}}{(1+\alpha-n/2)_{j_1}(1+\beta-n/2)_{j_2}}=\frac{\Gamma(n/2-\alpha)\Gamma(n/2-\beta)\Gamma(\alpha+\beta-n/2)}{\Gamma(n-\alpha-\beta)}$$

$$\times F_4(\alpha+\beta-n/2, 1+\alpha+\beta-n; 1+\alpha-n/2, 1+\beta-n/2; m_1^2/p^2, m_2^2/p^2), \tag{32}$$

where a hypergeometric function with two variables x, y is defined as $F_4(a,b;c,d;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{m!n!(c)_m(d)_n}x^m y^n$, which is called Appell's function. In this case the convergence condition is given as follows; $|m_1^2/p^2|^{\frac{1}{2}}+|m_2^2/p^2|^{\frac{1}{2}}<1$.[10]

(ii) The case of $\Gamma(-s)$ and $\Gamma(n/2-\beta-t)$

$\Gamma(n/2-\beta-t)$ has single poles at $t=n/2-\beta+j_2$ ($j_2=0,1,2,\ldots$), and $\Gamma(-s)$ does at $s=j_1$ as well as the case(i). Therefore the residue of this integral becomes as follows;

$$\begin{aligned}
&\mathrm{Residue}\Big[\mathrm{s}=\mathrm{j}_1, \mathrm{t}=\mathrm{n}/2-\beta+\mathrm{j}_2; \mathrm{Integrand}\Big]\\
&=\sum_{j_1}^{\infty}\sum_{j_2}^{\infty}\Big(-\frac{m_1}{p^2}\Big)^{j_1}\Big(-\frac{m_2}{p^2}\Big)^{j_2+\frac{n}{2}-\beta}\frac{\Gamma(\beta-n/2-j_2)\Gamma(n/2-\alpha-j_1)\Gamma(\alpha+j_1+j_2)}{j_1!(-1)^{j_1}j_2!(-1)^{j_2}\Gamma(n/2-\alpha-j_1-j_2)} \qquad (33)\\
&=\Gamma(\beta-n/2)\Gamma(\alpha)\Big(-\frac{m_2^2}{p^2}\Big)^{\frac{n}{2}-\beta}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\Big(\frac{m_1^2}{p^2}\Big)^{j_1}\Big(\frac{m_2^2}{p^2}\Big)^{j_2}\frac{(\alpha)_{j_1+j_2}(1+\alpha-n/2)_{j_1+j_2}}{j_1!j_2!(1+\alpha-n/2)_{j_1}(1+n/2-\beta)_{j_2}}\\
&=\Gamma(\beta-n/2)\Gamma(\alpha)\Big(-\frac{m_2^2}{p^2}\Big)^{\frac{n}{2}-\beta}F_4(\alpha, 1+\alpha-n/2; 1+\alpha-n/2, 1+n/2-\beta; m_1^2/p^2, m_2^2/p^2)
\end{aligned}$$

(iii) The case of $\Gamma(n/2-\alpha-s)$ and $\Gamma(-t)$

In this case the single poles are at $s=n/2-\alpha+j_1$ and $t=j_2$ ($j_1, j_2=0,1,2\ldots$). Then we can get the residue of the contour integral;

$$\begin{aligned}
&\mathrm{Residue}\Big[\mathrm{s}=\mathrm{n}/2-\alpha+\mathrm{j}_1, \mathrm{t}=\mathrm{j}_2; \mathrm{Integrand}\Big]\\
&=\sum_{j_1}^{\infty}\sum_{j_2}^{\infty}\Big(-\frac{m_1}{p^2}\Big)^{\frac{n}{2}-\alpha+j_1}\Big(-\frac{m_2}{p^2}\Big)^{j_2}\frac{\Gamma(\alpha-n/2-j_1)\Gamma(n/2-\beta-j_2)\Gamma(\beta+j_1+j_2)}{j_1!(-1)^{j_1}j_2!(-1)^{j_2}\Gamma(n/2-\beta-j_1-j_2)} \qquad (34)\\
&=\Gamma(\alpha-n/2)\Gamma(\beta)\Big(-\frac{m_1^2}{p^2}\Big)^{\frac{n}{2}-\alpha}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\Big(\frac{m_1^2}{p^2}\Big)^{j_1}\Big(\frac{m_2^2}{p^2}\Big)^{j_2}\frac{(\beta)_{j_1+j_2}(1+\beta-n/2)_{j_1+j_2}}{j_1!j_2!(1+n/2-\alpha)_{j_1}(1+\beta-n/2)_{j_2}}\\
&=\Gamma(\alpha-n/2)\Gamma(\beta)\Big(-\frac{m_1^2}{p^2}\Big)^{\frac{n}{2}-\alpha}F_4(\beta, 1+\beta-n/2; 1+n/2-\alpha, 1+\beta-n/2; m_1^2/p^2, m_2^2/p^2)
\end{aligned}$$

(iv) The case of $\Gamma(n/2-\alpha-s)$ and $\Gamma(n/2-\beta-t)$

In this case we recognize

$$\mathrm{Residue}\Big[\mathrm{s}=\mathrm{n}/2-\alpha+\mathrm{j}_1, \mathrm{t}=\mathrm{n}/2-\beta+\mathrm{j}_2; \mathrm{Integrand}\Big]=0, \tag{35}$$

because the denominator in the integrand becomes $\Gamma(n-\alpha-\beta-s-t)=\Gamma(n-\alpha-\beta-n/2+\alpha-j_1-n/2+\beta-j_2)=\Gamma(-j_1-j_2)=-\infty$, where j_1 and j_2 are integers from 0 to ∞. Finally we can obtain the result of integration (31) by adding up the residues of the cases (i),(ii),(iii) and (iv). Substituting (32), (33) and (34) for (31), the final result becomes as follows;

$$\begin{aligned}
&J(\alpha,\beta;m_1,m_2)=\pi^{\frac{n}{2}}i^{1-n}(p^2)^{\frac{n}{2}-\alpha-\beta}\\
&\times\Bigg[\frac{\Gamma(n/2-\alpha)\Gamma(n/2-\beta)\Gamma(\alpha+\beta-n/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-\alpha-\beta)}\\
&\times F_4(\alpha+\beta-n/2, \alpha+\beta-n+1; \alpha-n/2+1, 1+\beta-n/2; m_1^2/p^2, m_2^2/p^2) \qquad (36)\\
&+\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}\Big(-\frac{m_1^2}{p^2}\Big)^{\frac{n}{2}-\alpha}F_4(\beta, \beta-n/2+1; n/2-\alpha+1, \beta-n/2+1; m_1^2/p^2, m_2^2/p^2)\\
&+\frac{\Gamma(\beta-n/2)}{\Gamma(\beta)}\Big(-\frac{m_2^2}{p^2}\Big)^{\frac{n}{2}-\beta}F_4(\alpha, \alpha-n/2+1; \alpha-n/2+1, n/2-\beta+1; m_1^2/p^2, m_2^2/p^2)\Bigg],
\end{aligned}$$

where the convergence condition is $|m_1^2/p^2|^{\frac{1}{2}} + |m_2^2/p^2|^{\frac{1}{2}} < 1$.

Next we carry out a variable transformation $t = \frac{n}{2} - \alpha - \beta - s - u$.

We can get the following equation;

$$J(\alpha,\beta;m_1,m_2) = \pi^{\frac{n}{2}} i^{1-n}(-m_2^2)^{\frac{n}{2}-\alpha-\beta}[\Gamma(\alpha)\Gamma(\beta)]^{-1}\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} dsdu\Big(\frac{m_1^2}{m_2^2}\Big)^s\Big(-\frac{p^2}{m_2^2}\Big)^u$$
$$\times\frac{\Gamma(-s)\Gamma(-u)\Gamma(n/2-\alpha-s)\Gamma(\alpha+\beta-n/2+s+u)\Gamma(\alpha+s+u)}{\Gamma(n/2+u)} \quad (37)$$

In this case there exist the two ways to calculate the residue by superimposing s complex sheet on u complex sheet and closing the contours on their right half planes. (i) the case of considering single poles from $\Gamma(-s)$ and $\Gamma(-u)$, (ii) the case of considering single poles from $\Gamma(n/2-\alpha-s)$ and $\Gamma(-u)$. The calculations of the residues are performed as well as above procedure.

(i) the case of $\Gamma(-s)$ and $\Gamma(-u)$

$$\mathrm{Residue}[\mathrm{s}=\mathrm{j}_1,\mathrm{u}=\mathrm{j}_2;\mathrm{Integrand}] = \sum_{\mathrm{j}_1=0}^{\infty}\sum_{\mathrm{j}_2=0}^{\infty}\Big(\frac{\mathrm{m}_1^2}{\mathrm{m}_2^2}\Big)^{\mathrm{j}_1}\Big(-\frac{\mathrm{p}^2}{\mathrm{m}_2^2}\Big)^{\mathrm{j}_2}$$
$$\times\frac{\Gamma(n/2-\alpha-j_1)\Gamma(\alpha+\beta-n/2+j_1+j_2)\Gamma(\alpha+j_1+j_2)}{j_1!(-1)^{j_1}j_2!(-1)^{j_2}\Gamma(n/2+j_2)} \quad (38)$$
$$=\frac{\Gamma(n/2-\alpha)\Gamma(\alpha)\Gamma(\alpha+\beta-n/2)}{\Gamma(n/2)}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\Big(\frac{m_1^2}{m_2^2}\Big)^{j_1}\Big(\frac{p^2}{m_2^2}\Big)^{j_2}\frac{(\alpha)_{j_1+j_2}(\alpha+\beta-n/2)_{j_1+j_2}}{j_1!j_2!(n/2)_{j_2}(1+\alpha-n/2)_{j_1}}$$
$$=\frac{\Gamma(n/2-\alpha)\Gamma(\alpha)\Gamma(\alpha+\beta-n/2)}{\Gamma(n/2)}F_4(\alpha,\alpha+\beta-n/2;1+\alpha-n/2,n/2;m_1^2/m_2^2,p^2/m_2^2),$$

(ii) The case of $\Gamma(n/2-\alpha-s)$ and $\Gamma(-u)$

$$\mathrm{Residue}\Big[\mathrm{s}=-\alpha+\frac{\mathrm{n}}{2}+\mathrm{j}_1,\mathrm{u}=\mathrm{j}_2;\mathrm{Integrand}\Big]$$
$$=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\Big(\frac{m_1^2}{m_2^2}\Big)^{-\alpha+\frac{n}{2}+j_1}\Big(-\frac{p^2}{m_2^2}\Big)^{j_2}\frac{\Gamma(\alpha-n/2-j_1)\Gamma(\beta+j_1+j_2)\Gamma(n/2+j_1+j_2)}{j_1!(-1)^{j_1}j_2!(-1)^{j_2}\Gamma(n/2+j_2)}$$
$$=\Big(\frac{m_1^2}{m_2^2}\Big)^{\frac{n}{2}-\alpha}\Gamma(\alpha-n/2)\Gamma(\beta)\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\frac{(\beta)_{j_1+j_2}(n/2)_{j_1+j_2}}{j_1!j_2!(n/2)_{j_2}(1+n/2-\alpha)_{j_1}}\Big(\frac{m_1^2}{m_2^2}\Big)^{j_1}\Big(\frac{p^2}{m_2^2}\Big)^{j_2}$$
$$=\Big(\frac{m_1^2}{m_2^2}\Big)^{\frac{n}{2}-\alpha}\Gamma(\alpha-n/2)\Gamma(\beta)F_4(\beta,n/2;1+n/2-\alpha,n/2;m_1^2/m_2^2,p^2/m_2^2) \quad (39)$$

From the results of the cases (i),(ii) we have

$$J(\alpha,\beta;m_1,m_2) = \pi^{\frac{n}{2}} i^{1-n}(-m_2^2)^{\frac{n}{2}-\alpha-\beta}$$
$$\times\Bigg[\frac{\Gamma(n/2-\alpha)\Gamma(\alpha+\beta-n/2)}{\Gamma(n/2)\Gamma(\beta)}F_4(\alpha,\alpha+\beta-n/2;1+\alpha-n/2,n/2;m_1^2/m_2^2,p^2/m_2^2)$$
$$+\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}\Big(\frac{m_1^2}{m_2^2}\Big)^{\frac{n}{2}-\alpha}F_4(\beta,n/2;1-\alpha+n/2,n/2;m_1^2/m_2^2,p^2/m_2^2)\Bigg]. \quad (40)$$

We can write down an analytic continuation formula with respect to hypergeometric function $F_4(x,y)$ by considering the relation between (36) and (40).

Taking assignment as $m_1^2/p^2 = x, m_2^2/p^2 = y, n = 4, \alpha = a, \beta = b$, we have

$$\frac{\Gamma(2-a)\Gamma(a+b-2)}{\Gamma(b)} F_4\left(a, a+b-2; a-1, 2; \frac{x}{y}, \frac{1}{y}\right) + \frac{\Gamma(a-2)}{\Gamma(a)}\left(\frac{x}{y}\right)^{2-a} F_4\left(b, 2; 3-a, 2; \frac{x}{y}, \frac{1}{y}\right)$$
$$= (-y)^{a+b-2}\left[\frac{\Gamma(2-a)\Gamma(2-b)\Gamma(a+b-2)}{\Gamma(a)\Gamma(b)\Gamma(4-a-b)} F_4\Big(a+b-2, a+b-3; a-1, b-1; x, y\Big) \right. \tag{41}$$
$$\left. + \frac{\Gamma(a-2)}{\Gamma(a)}\Big(-x\Big)^{2-a} F_4\Big(b, b-1; 3-a, b-1; x, y\Big) + \frac{\Gamma(b-2)}{\Gamma(b)}\Big(-y\Big)^{2-b} F_4\Big(a, a-1; a-1, 3-b; x, y\Big)\right].$$

5 Concluding remarks

In this paper we introduced and reviewed the Davydychev method concerning the calculation of loop type Feynman integral. Then we found new analytic continuation formulas on hypergeometric functions ${}_3F_2(a, b, c; d, e; z)$ and $F_4(a, b; c, d; x, y)$. The Davydychev method exploiting hypergeometric functions has the following merits. (i) We can investigate analytic properties of Feynman propagators by examining hypergeometric functions. (ii) We can make several new analytic continuation formulas or recurrence ones of several hypergeometric functions using the results of Feynman propagator integral calculations. The demerits are as follows, (i) The calculations are more complex rather than usual ones. (ii) Because the knowledges of hypergeometric functions are still inadequate, it is difficult to calculate more complex Feynman integrals by using hypergeometric function theory.

In this paper we calculated the simplest loop type Feynman integral, but we have to investigate more complex Feynman integrals by making use of hypergeometric functions. Then we can make many new analytic continuation formulas or recurrence ones on many hypergeometric functions by performing the calculations of more complex Feynman integrals and contribute to the development of hypergeometric function theory.

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